量子光学
第一章 辐射场及其量子化

黄月新

大湾区大学

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课程大纲

量子光学是研究光场的量子性质，及其与原子、分子等物质相互作用的物理学分支，核心内容包括光子的产生、操控与探测以及光–物质的量子相干与纠缠现象。

第一章 辐射场及其量子化 (2 次课)

第二章 **量子相干态** (3 次课)

第三章 **光场相干性及其干涉** (3 次课)

第四章 **光场与原子相互作用** (4 次课)

第五章 热库系统与主方程 (2 次课)

第六章 激光冷却技术 (2 次课)

第七章 光学腔和光力耦合系统 (2 次课)

参考资料

- 《量子光学》，郭光灿，周祥发，科学出版社
- Quantum Optics, Marlan O. Scully, M. Suhail Zubairy, Cambridge University Press
- Lecture of Professor Farhan Rana
- 维基百科 (wikipedia.org)

课程

课程主页:

https://gbuphys.team/Quantum_Optics/QuantumOptics.html

考核要求:

- 平时成绩 (10%)
- 作业 (40%)
- 期末考试 (50%)

第一章

量子光学简介

量子力学公设

谐振子量子化

电磁场及其量子化

热平衡光场

量子相位算符

总结

第一章

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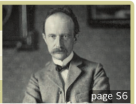
量子相位算符

总结

光学发展

MILESTONES TIMELINE

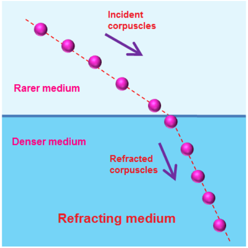
1600s–1800s	Debate on the character of light (Milestone 1)
1861	Maxwell's equations (Milestone 2)
1900	Planck's theory of black-body radiation (Milestone 3)
1905	Special relativity (Milestone 4)
1923	Compton effect (Milestone 5)
1947	Quantum electrodynamics (Milestone 6)
1948	Holograms (Milestone 7)
1954	Solar cells (Milestone 8)
1960	The laser (Milestone 9)
1961	Nonlinear optics (Milestone 10)
1963	Quantum optics (Milestone 11)
1964	Bell inequality (Milestone 12)
1966	Optical fibres (Milestone 13)
1970	CCD cameras (Milestone 14)
	Semiconductor lasers (Milestone 15)
1981	High-resolution laser spectroscopy and frequency metrology (Milestone 16)
1982–1985	Quantum information (Milestone 17)
1987	Photonic crystals (Milestone 18)
1993	Blue light-emitting diodes (Milestone 19)
1998	Plasmonics (Milestone 20)
2000	Metamaterials (Milestone 21)
2001	Attosecond science (Milestone 22)
2006	Cavity optomechanics (Milestone 23)



S4 | MAY 2010 www.nature.com/milestones/photons

光的“微粒说”(corpuscular theory of light)

Light is made up of small discrete particles called “corpuscles” which travel in a straight line



- 牛顿 1643-1727
- 直线传播、反射、折射
- 18 世纪时，微粒说是光的主流理论

光的“波动说”(light wave theory)

Huygens’ Construction

Huygens' construction is a method developed by Christiaan Huygens to explain how wavefronts propagate.

- *Huygens proposed that every point on a wavefront acts as a source of new waves.*
- *These new waves, called wavelets, spread out in spheres at the speed of light.*
- *The new wavefront is then formed by the envelope of these wavelets.*

- 胡克（17 世纪）
- 惠更斯 (17 世纪)
- 杨氏双缝干涉实验
- 菲涅尔 (18 世纪早期)
- 麦克斯韦 (18 世纪中后期)
- 波动说可以解释：干涉、散射、极化

波粒二象性

- 普朗克 (1858-1947): 光场能量量子化
- 爱因斯坦 (1879-1955): 光电效应

光可以同时具有粒子性和波动性

- 德布罗意 (1892-1987): 物质波假说

光场量子特性

- 20 世纪 60 年代激光技术发现以后，人们才开始研究光的量子特性

亮度高、单色性好、单向、相干性强

- 1956 年，首次观测到光场的强度关联效应 (HBT 干涉效应)
- 量子光学着重研究光子与原子及其它微观量子系统之间的关联和相互作用等各种量子效应

量子光学在量子力学的地位

量子光学研究

- 相干态
- Fock 态
- 纠缠态
- 压缩态
- 量子测量

量子光学在量子力学的地位

量子光学研究

- 相干态
- Fock 态
- 纠缠态
- 压缩态
- 量子测量

很多量子力学的概念

- 波函数塌缩
- 量子非定域性
- 贝尔不等式

都是在量子光学实验中首次严格验证

同时，量子光学又为量子力学的应用提供技术手段

量子光学有什么用？

- 既是理论验证平台，又是工程技术核心
- 量子信息、量子计算
- 重要的技术手段：冷却、原子操控、探测

  玻色爱因斯坦凝聚态、引力波探测、原子量子态实现
- 学科交叉：量子光学连接

  原子物理、分子物理、冷原子、固体量子器件

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总结

量子力学公设

- 波函数公设
- 算符公设
- 薛定谔方程公设

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi$$

- 测量公设

$$\bar{A} = \frac{\int \mathrm{d}\boldsymbol{r}\ \langle\psi|A|\psi\rangle}{\int \mathrm{d}\boldsymbol{r}\,\langle\psi|\psi\rangle}$$

- 全同性原理

波函数公设

一个系统在某个时刻的状态可以由一个复函数完全表示 $\psi(\boldsymbol{r}, t)$

- $|\psi\rangle$ 所属的空间称为希尔伯特空间:linear vector space + metric
- $|\psi_k\rangle \sim |\psi_l\rangle \leftrightarrow |\psi_k\rangle = e^{i\alpha} |\psi_l\rangle$
- $|\psi(\boldsymbol{r}, t)|^2$ 为对应位置和时间的概率

$$\int |\psi(\boldsymbol{r}, t)|^2 \, \mathrm{d}\boldsymbol{r} = 1$$

- 叠加态 (superposition)

$$c_1 |\psi_1\rangle + c_2 |\psi_2\rangle$$

c_1 和 c_2 为复数，满足 $|c_1|^2 + |c_2|^2 = 1$

例子：Qubit and Bloch sphere

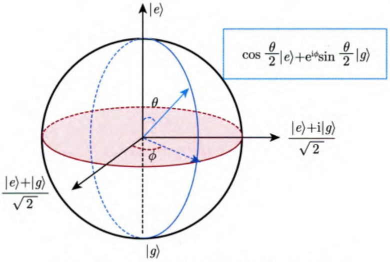

图 1.1　二维量子态及对应的 Bloch 矢量

- 具有两个能级的量子态

$$|\psi\rangle = \alpha\,|g\rangle + \beta\,|e\rangle$$

- ψ 和 $\lambda\psi$ 代表相同的状态：projective Hilbert space $P(\mathcal{H}_2) = CP^1$
- α 和 β 为复数，意味着可以用四个实数来表示一个 Qubit，然而只有相对的相位是有意义的，因此实际上只需要三个实数
- 对于纯态，由于归一化条件，每个纯态都落在球面上，因此

$$|\psi\rangle = \cos\frac{\theta}{2}\,|g\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|e\rangle$$

例子：Qubit and Bloch sphere

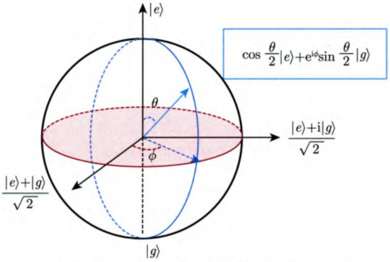

图 1.1　二维量子态及对应的 Bloch 矢量

- 密度矩阵

$$\rho = \sum_n p_n \, |\psi_n\rangle\langle\psi_n|$$

- ρ 是厄米的，可以用泡利矩阵表示

$$\rho = \frac{1}{2}(I + \boldsymbol{R}\cdot\boldsymbol{\sigma}) = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{R_x}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \frac{R_y}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + \frac{R_z}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

- $|R|^2 = R_x^2 + R_y^2 + R_z^2 = 1 - 4(\rho_{ee}\rho_{gg} - \rho_{eg}\rho_{ge})$

例子：Qubit and Bloch sphere

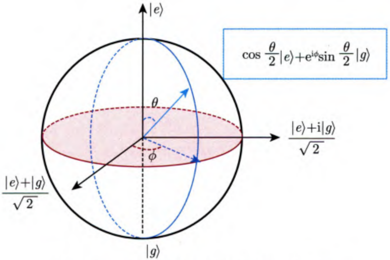

图 1.1　二维量子态及对应的 Bloch 矢量

- 纯态密度矩阵的秩为 1→ $\det(\rho_{\mathrm{pure}}) = 0$

$$\rho_{gg}\rho_{ee} - \rho_{eg}\rho_{ge} = 0, \quad |R| = 1$$

- 对于混态 $0 < \det(\rho) < 1$，因此 $|R| < 1$

混态在 Bloch 球内部

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复合系统和纠缠态

假定有 A、B 两个小系统，它们各自对应的希尔伯特空间记为 $\mathcal{H}_A$、$\mathcal{H}_B$，其中一组正交完备基矢为

$$\{|a_m\rangle\,,\ 1 \le m \le M\}\,,\quad \{|b_n\rangle\,,\ 1 \le n \le N\}$$

复合系统对应的希尔伯特空间为 $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$，它的基矢为

$$\{|e_{mn}\rangle = |a_m\rangle \otimes |b_n\rangle\,,\ 1 \le m \le M,\ 1 \le n \le N\}$$

- 不同本征态 $|a_1\rangle\,|b_1\rangle$ 和 $|a_2\rangle\,|b_2\rangle$ 的叠加

$$|\psi\rangle_{12} \sim |a_1\rangle\,|b_1\rangle + |a_2\rangle\,|b_2\rangle$$

- 上述叠加态不可能写成 A、B 各自子系统中某个状态 $|\phi\rangle_A$ 和 $|\eta\rangle_B$ 直积的形式

$$|\psi\rangle_{12} \neq |\phi\rangle_A \otimes |\eta\rangle_B$$

这样的态被称为纠缠态

复合系统和纠缠态

- **贝尔态** (Bell state)

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

复合系统和纠缠态

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$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

- GHZ 态

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

复合系统和纠缠态

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- GHZ 态

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

- W 态

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

复合系统和纠缠态

- 贝尔态 (Bell state)

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- GHZ 态

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

- W 态

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right)$$

- NOON 态

$$|\mathrm{NOON}\rangle = \frac{1}{\sqrt{2}}\left(|N\rangle_A |0\rangle_B + |0\rangle_A |N\rangle_B\right)$$

复合系统和纠缠态

- **以下量子态是否可以称为纠缠态**

$$|\psi\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

复合系统和纠缠态

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$$|\psi\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

可以发现

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)_A \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)_B$$

复合系统和纠缠态

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不是纠缠态

复合系统和纠缠态

对于两体系统，任意纯态可以写成

$$|\psi\rangle = \sum_{m,n} c_{mn} |a_m\rangle |b_n\rangle$$

- c_{mn} 为矩阵 C 的元素，维度为 $M \times N$
- 奇异值分解 (singular value decomposition)

$$C = U \cdot D \cdot V$$

- ψ 可以在新的基矢下展开

$$\begin{aligned}
|\psi\rangle &= \sum_{m,n} \sum_{k} U_{mk} D_{kk} V_{kn} |a_m\rangle |b_n\rangle \\
&= \sum_{k} D_{kk} \left(\sum_{m} U_{mk} |a_m\rangle \right) \otimes \left(\sum_{n} V_{kn} |b_n\rangle \right) \\
&= \sum_{k} D_{kk} |k\rangle_A \otimes |k\rangle_B
\end{aligned}$$

- 如果 D_{kk} 只有一个非零元素，ψ 为直积态
- 如果 D_{kk} 有大于一个非零元素，ψ 为纠缠态

复合系统和纠缠态

- 如果存在多个不可分割的态叠加，则每个组分越均匀，纠缠度越高
- 两组分系统，四个最大纠缠态

$$
\begin{aligned}
\left|\phi^{+}\right\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle_A\,|0\rangle_B + |1\rangle_A\,|1\rangle_B\right) \\
\left|\phi^{-}\right\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle_A\,|0\rangle_B - |1\rangle_A\,|1\rangle_B\right) \\
\left|\psi^{+}\right\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle_A\,|1\rangle_B + |1\rangle_A\,|0\rangle_B\right) \\
\left|\psi^{-}\right\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle_A\,|1\rangle_B - |1\rangle_A\,|0\rangle_B\right)
\end{aligned}
$$

以上这些都称为贝尔态，在量子信息和量子计算中有着极其重要的作用

薛定谔方程与表象

$$i\hbar\partial_t |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

- **幺正变换** $|\psi(t)\rangle = \hat{U} |\phi(t)\rangle$，**则哈密顿量变为**

$$i\hbar\partial_t\hat{U} |\phi(t)\rangle = \hat{H}\hat{U} |\phi(t)\rangle$$

薛定谔方程与表象

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$$i\hbar\partial_t\hat{U} |\phi(t)\rangle = \hat{H}\hat{U} |\phi(t)\rangle$$

- 若 $\hat{U}$ 不含时，上式两边乘以 $\hat{U}^\dagger$

$$i\hbar\partial_t |\phi(t)\rangle = \hat{U}^\dagger\hat{H}\hat{U} |\phi(t)\rangle$$

- 有效哈密顿量为

$$H_{\text{eff}} = \hat{U}^\dagger\hat{H}\hat{U}$$

薛定谔方程与表象

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$$i\hbar\partial_t\hat{U} |\phi(t)\rangle = \hat{H}\hat{U} |\phi(t)\rangle$$

- 若 $\hat{U}(t)$ 是含时的，

$$i\hbar\hat{U}^\dagger(t)\left[\partial_t\hat{U}\right]|\phi(t)\rangle + i\hbar\hat{U}^\dagger(t)\hat{U}(t)\partial_t |\phi(t)\rangle = \hat{U}^\dagger(t)\hat{H}\hat{U}(t) |\phi(t)\rangle$$

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- 若 $\hat{U}(t)$ 是含时的，

$$i\hbar\partial_t |\phi(t)\rangle = \left\{\hat{U}^\dagger(t)\hat{H}\hat{U}(t) - i\hbar\hat{U}^\dagger(t)\left[\partial_t\hat{U}(t)\right]\right\} |\phi(t)\rangle$$

薛定谔方程与表象

$$i\hbar\partial_t |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

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薛定谔方程与表象

$$\hat{H}_{\text{eff}} = \hat{U}^\dagger(t)\hat{H}\hat{U}(t) - i\hbar\hat{U}^\dagger(t)\left[\partial_t\hat{U}(t)\right]$$

- 海森堡表象 (绘景): 若 $\hat{U}(t) = \exp\left(-\frac{i}{\hbar}Ht\right)$

$$\hat{H}_{\text{eff}} = 0$$

- 所有算符在新的表象下

$$\hat{A} \to \hat{U}^\dagger\hat{A}\hat{U}$$

$$\hat{H}_{\text{eff}} = \hat{U}^\dagger(t)\hat{H}\hat{U}(t) - i\hbar\hat{U}^\dagger(t)\left[\partial_t\hat{U}(t)\right]$$

- **相互作用表象 (绘景):** 若 $\hat{U}(t) = \exp\left(-\frac{i}{\hbar}H_0 t\right)$

$$\begin{aligned}\hat{H}_{\text{eff}} &= \exp\left(\frac{i}{\hbar}\hat{H}_0 t\right)(\hat{H}_0 + \hat{V})\exp\left(-\frac{i}{\hbar}\hat{H}_0 t\right) - i\hbar\exp\left(\frac{i}{\hbar}\hat{H}_0 t\right)\partial_t\exp\left(-\frac{i}{\hbar}\hat{H}_0 t\right)\\ &= \exp\left(\frac{i}{\hbar}\hat{H}_0 t\right)\hat{V}\exp\left(-\frac{i}{\hbar}\hat{H}_0 t\right)\end{aligned}$$

例子 1

二能级系统

$$H = \omega_1 |1\rangle\langle 1| + \omega_2 |2\rangle\langle 2| + \Omega |1\rangle\langle 2| + \Omega |2\rangle\langle 1|$$

例子 1

二能级系统

$$H = \begin{pmatrix} \omega_1 & \Omega \\ \Omega & \omega_2 \end{pmatrix}$$

例子 1

二能级系统

$$H = \begin{pmatrix} \omega_1 & \Omega \\ \Omega & \omega_2 \end{pmatrix}$$

- 对角化 $\hat{H}$

$$\hat{H}\hat{U} = \hat{U}\begin{pmatrix} -\sqrt{\Delta\omega^2+\Omega^2} & \\ & \sqrt{\Delta\omega^2+\Omega^2} \end{pmatrix}$$

其中 $\Delta\omega = \omega_2 - \omega_1$

- v_1 和 v_2 为正交归一化的本征态，$\hat{U} = (v_1, v_2)$
- 时间演化

$$\begin{aligned} |\psi(t)\rangle &= (|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|)\,|\psi(t)\rangle \\ &= c_1 e^{i\sqrt{\Delta\omega^2+\Omega^2}t}\,|v_1\rangle + c_2 e^{-i\sqrt{\Delta\omega^2+\Omega^2}t}\,|v_2\rangle \end{aligned}$$

$c_1 = \langle v_1|\psi(0)\rangle$, $c_2 = \langle v_2|\psi(0)\rangle$ 为初态在本征态上的投影系数

例子 2

二能级系统

$$H = \Delta\omega \left|2\right\rangle\left\langle 2\right| + \Omega e^{i\omega_c t} \left|1\right\rangle\left\langle 2\right| + \Omega e^{-i\omega_c t} \left|2\right\rangle\left\langle 1\right|$$

设 $\hat{U} = \exp(-i\omega_c t \left|2\right\rangle\left\langle 2\right|)$, **则**

$$\hat{H}_{\text{eff}} = \hat{U}^\dagger \hat{H} \hat{U} - i\hbar \hat{U}^\dagger \partial_t \hat{U}$$

- **利用公式**

$$e^{\alpha\hat{A}} \hat{B} e^{-\alpha\hat{A}} = \hat{B} + \alpha\left[\hat{A}, \hat{B}\right] + \frac{\alpha^2}{2!}\left[\hat{A}, [\hat{A}, \hat{B}]\right] + \ldots$$

求

$$\begin{aligned}
& e^{i\omega_c t|2\rangle\langle 2|} \left|1\right\rangle\left\langle 2\right| e^{-i\omega_c t|2\rangle\langle 2|} \\
= & \left|1\right\rangle\left\langle 2\right| + i\omega_c t[\left|2\right\rangle\left\langle 2\right|, \left|1\right\rangle\left\langle 2\right|] + \frac{(i\omega_c t)^2}{2!}[\left|2\right\rangle\left\langle 2\right|, [\left|2\right\rangle\left\langle 2\right|, \left|1\right\rangle\left\langle 2\right|]] + \ldots \\
= & \left|1\right\rangle\left\langle 2\right| - i\omega_c t \left|1\right\rangle\left\langle 2\right| + \frac{(i\omega_c t)^2}{2!} \left|1\right\rangle\left\langle 2\right| + \ldots \\
= & e^{-i\omega_c t} \left|1\right\rangle\left\langle 2\right|
\end{aligned}$$

例子 2

二能级系统

$$H = \Delta\omega \left|2\right\rangle\!\left\langle 2\right| + \Omega e^{i\omega_c t} \left|1\right\rangle\!\left\langle 2\right| + \Omega e^{-i\omega_c t} \left|2\right\rangle\!\left\langle 1\right|$$

设 $\hat{U} = \exp(-i\omega_c t \left|2\right\rangle\!\left\langle 2\right|)$，**则**

$$\hat{H}_{\text{eff}} = \hat{U}^\dagger \hat{H} \hat{U} - i\hbar \hat{U}^\dagger \partial_t \hat{U}$$

- **有效哈密顿量为**

$$\hat{H}_{\text{eff}} = (\Delta\omega - \omega_c) \left|2\right\rangle\!\left\langle 2\right| + \Omega \left|1\right\rangle\!\left\langle 2\right| + \Omega \left|2\right\rangle\!\left\langle 1\right|$$

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总结

谐振子量子化

一维谐振子

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

- 动量算符 $\hat{p} = -i\hbar\partial_x$，满足 $[\hat{x}, \hat{p}] = i\hbar$
- 薛定谔方程

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2\right)\psi(x) = E_n\psi(x)$$

严格可解的二阶微分方程

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

- **定义算符**

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}} (m\omega\hat{x} + i\hat{p}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}} (m\omega\hat{x} - i\hat{p})$$

- **利用** $[\hat{x}, \hat{p}] = i\hbar$，**可以得到**

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1$$

- **把** $\hat{x}$ **和** $\hat{p}$ **代入** H

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \quad \hat{p} = -\sqrt{\frac{m\hbar\omega}{2}}(a - a^\dagger)$$

得到

$$H = \frac{1}{2}\left(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

谐振子量子化
一维谐振子

$$H = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

- 假定谐振子的本征态 $|n\rangle$，对应本征能量为 E_n

$$H|n\rangle = E_n|n\rangle$$

- 对易关系

$$\left[\hat{a}, \hat{H}\right] = \left[\hat{a}, \hbar\omega\hat{a}^\dagger\hat{a}\right] = \hbar\omega\hat{a}, \quad \left[\hat{a}^\dagger, \hat{H}\right] = \left[\hat{a}^\dagger, \hbar\omega\hat{a}^\dagger\hat{a}\right] = -\hbar\omega\hat{a}^\dagger$$

可得

$$\hat{H}\hat{a}|n\rangle = \left(\hat{a}\hat{H} - \hbar\omega\hat{a}\right)|n\rangle = (E_n - \hbar\omega)\,\hat{a}|n\rangle,$$
$$\hat{H}\hat{a}^\dagger|n\rangle = \left(\hat{a}^\dagger\hat{H} + \hbar\omega\hat{a}^\dagger\right)|n\rangle = (E_n + \hbar\omega)\,\hat{a}^\dagger|n\rangle$$

1. $\hat{a}|n\rangle$ 是本征态，对应本征能量为 $E_n - \hbar\omega$
2. $\hat{a}^\dagger|n\rangle$ 是本征态，对应本征能量为 $E_n + \hbar\omega$

谐振子量子化

- $\hat{a}^\dagger$ 产生能量 $\hbar\omega$, $\hat{a}$ 湮灭能量 $\hbar\omega$ $\rightarrow$

  产生算符 $\hat{a}^\dagger$、湮灭算符 $\hat{a}$

- 将 $\hat{a}$ 不断作用到 $|n\rangle$ 上，由于谐振子能量不可能小于 0，当 $\hat{a}$ 作用到基态上时，不会有新的状态出现，因此

$$\hat{a}\,|0\rangle = 0$$

- 基态能量：$\hat{H}\,|0\rangle = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|0\rangle = \frac{1}{2}\hbar\omega$
- 算符作用：假定

$$\hat{a}\,|n\rangle = \alpha_n\,|n-1\rangle\,, \quad \hat{a}^\dagger\,|n\rangle = \beta_n\,|n+1\rangle$$

谐振子量子化

- 由 $\hat{a}^\dagger \hat{a} |n\rangle = \hat{a}^\dagger \alpha_n |n-1\rangle = \beta_{n-1}\alpha_n |n\rangle$ 可以得到

$$\beta_{n-1}\alpha_n = n$$

- 由于 $\hat{a}$ 是厄米算符 $\rightarrow \alpha_n = \beta_{n-1}^*$

谐振子量子化

- 由 $\hat{a}^\dagger \hat{a} |n\rangle = \hat{a}^\dagger \alpha_n |n-1\rangle = \beta_{n-1}\alpha_n |n\rangle$ 可以得到

$$\beta_{n-1}\alpha_n = n$$

- 由于 $\hat{a}$ 是厄米算符 $\rightarrow \alpha_n = \beta_{n-1}^*$

一般，可以认为 α_n 和 β_n 为实数，所以 $\alpha_n = \sqrt{n}$, $\beta_n = \sqrt{n+1}$

谐振子量子化

- 由 $\hat{a}^\dagger\hat{a}\left|n\right\rangle = \hat{a}^\dagger\alpha_n\left|n-1\right\rangle = \beta_{n-1}\alpha_n\left|n\right\rangle$ 可以得到

$$\beta_{n-1}\alpha_n = n$$

- 由于 $\hat{a}$ 是厄米算符 $\to \alpha_n = \beta^*_{n-1}$

最后，得到

$$\hat{a}\left|0\right\rangle = 0, \quad \hat{a}^\dagger\hat{a}\left|n\right\rangle = n\left|n\right\rangle$$
$$\hat{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle, \quad \hat{a}^\dagger\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle$$

本征态

$$|n\rangle \propto \left(\hat{a}^{\dagger}\right)^{n} |0\rangle$$

- 归一化

$$\langle 0|\hat{a}^{n} \left(\hat{a}^{\dagger}\right)^{n} |0\rangle = n!$$

谐振子量子化

本征态

$$|n\rangle = \frac{\left(\hat{a}^{\dagger}\right)^{n}}{\sqrt{n!}}\,|0\rangle$$

谐振子量子化

- $|n\rangle$ 为粒子数算符 $\hat{N} = \hat{a}^\dagger \hat{a}$ 的本征态
- Fock 态表象，粒子数表象

$$\hat{a}_{mn} = \langle m|\hat{a}|n\rangle = \sqrt{n}\delta_{m,n-1}, \quad \hat{a}^\dagger_{mn} = \sqrt{n+1}\delta_{m,n+1}$$

矩阵形式

$$\hat{a} = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \\ \vdots & \vdots & \vdots & \ldots & \end{pmatrix}$$

$$\hat{a}^\dagger = {\color{red}?}$$

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经典电磁场

无源条件下

$$\nabla \cdot \boldsymbol{B} = 0$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \cdot \boldsymbol{D} = 0$$

$$\nabla \times \boldsymbol{H} = \frac{\partial D}{\partial t}$$

其中 $\boldsymbol{B} = \mu_0 \boldsymbol{H}$, $\boldsymbol{D} = \epsilon_0 \boldsymbol{E}$, $c^2 = 1/(\epsilon_0 \mu_0)$

- $\boldsymbol{E}$: 电场
- $\boldsymbol{D}$: 电位移矢量
- $\boldsymbol{B}$: 磁通量密度
- $\boldsymbol{H}$: 磁场强度

经典电磁场

- 引入矢量势 $\boldsymbol{A}$ 和标量势 Φ

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \quad \boldsymbol{E} = -\nabla\Phi - \frac{\partial \boldsymbol{A}}{\partial t}$$

- 给定一个解析函数

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla f, \quad \Phi' = \Phi - \frac{1}{c}\frac{\partial f}{\partial t}$$

  $(\boldsymbol{A}, \Phi)$ 和 $(\boldsymbol{A}', \Phi')$ 都对应同样的场强 $(\boldsymbol{E}, \boldsymbol{B})$

- 库仑规范 $\nabla \cdot \boldsymbol{A} = 0$

经典电磁场

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\frac{\partial}{\partial t}(\nabla \times \boldsymbol{B})$$

$$\Longrightarrow \nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = \mu_0 \frac{\partial}{\partial t} \frac{\partial}{\partial t} \boldsymbol{D}$$

$$\nabla^2 \boldsymbol{E} - \frac{1}{c^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0$$

- 电场的波动方程

经典电磁场

$$\nabla \times (\nabla \times \boldsymbol{H}) = \frac{\partial}{\partial t}(\nabla \times \boldsymbol{D})$$

$$\Longrightarrow \frac{1}{\mu_0}\nabla(\nabla \cdot \boldsymbol{B}) - \frac{1}{\mu_0}\nabla^2 \boldsymbol{B} = -\epsilon_0 \frac{\partial}{\partial t}\frac{\partial}{\partial t}\boldsymbol{B}$$

$$\nabla^2 \boldsymbol{B} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{B}}{\partial t^2} = 0$$

- 磁场的波动方程

经典电磁场

由 $\nabla \times \boldsymbol{H} = \frac{\partial D}{\partial t}$ 可得

$$
\begin{aligned}
\nabla \times \frac{1}{\mu_0}(\nabla \times \boldsymbol{A}) &= \epsilon_0 \frac{\partial}{\partial t}\left(-\nabla\Phi - \frac{\partial \boldsymbol{A}}{\partial t}\right) \\
\rightarrow \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} &= \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla\Phi - \frac{\partial \boldsymbol{A}}{\partial t}\right) \\
\rightarrow \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{A} &= \nabla\left(\frac{1}{c^2}\frac{\partial \Phi}{\partial t} + \nabla \cdot \boldsymbol{A}\right)
\end{aligned}
$$

经典电磁场

由 $\nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t}$ 可得

$$\nabla \times \frac{1}{\mu_0}(\nabla \times \boldsymbol{A}) = \epsilon_0 \frac{\partial}{\partial t}\left(-\nabla \Phi - \frac{\partial \boldsymbol{A}}{\partial t}\right)$$

$$\to \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla \Phi - \frac{\partial \boldsymbol{A}}{\partial t}\right)$$

$$\to \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{A} = \nabla\left(\frac{1}{c^2}\frac{\partial \Phi}{\partial t} + \nabla \cdot \boldsymbol{A}\right)$$

选库仑规范 $\nabla \cdot \boldsymbol{A} = 0$，由 $\nabla \cdot \boldsymbol{E} = 0$ 可以得到

$$\nabla^2 \boldsymbol{\Phi} = -\frac{\partial}{\partial t}\nabla \cdot \boldsymbol{A}$$

因此在无源场下，可以令 $\Phi = 0$。从而，得到以下方程

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0$$

- Wave equation

只需要解三个方程之一就可以

谐振腔内电磁场的量子化

腔内电磁场满足的方程

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0$$

- 变量分离法
- 认为电场是许多本征模式的叠加

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{1}{\sqrt{\epsilon_0}} \sum_m q_m(t) \boldsymbol{u}_m(\boldsymbol{r})$$

- $\boldsymbol{u}_m(\boldsymbol{r})$ 为本征模式、$q_m(t)$ 为随时间变化的系数

$$\nabla^2 \boldsymbol{u}_m(\boldsymbol{r}) + k_m^2 \boldsymbol{u}_m(\boldsymbol{r}) = 0, \quad \frac{\mathrm{d}^2 q_m}{\mathrm{d}t^2} + c^2 k_m^2 q_m = 0$$

- 本征模式满足正交归一性

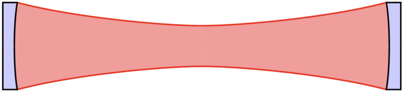
$$\int \boldsymbol{u}_m(\boldsymbol{r}) \boldsymbol{u}_n(\boldsymbol{r}) \,\mathrm{d}V = \delta_{mn}$$

- 得到电场和磁场

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} = -\frac{1}{\sqrt{\epsilon_0}} \sum_m \dot{q}_m(t) \boldsymbol{u}_m(\boldsymbol{r})$$

$$\boldsymbol{H} = \frac{1}{\mu_0} \nabla \times \boldsymbol{A} = \frac{c}{\sqrt{\mu_0}} \sum_m q_m(t) \nabla \times \boldsymbol{u}_m(\boldsymbol{r})$$

谐振腔内电磁场的量子化



$$\nabla^2 \boldsymbol{u}_l(\boldsymbol{r}) + k_l^2 \boldsymbol{u}_l(\boldsymbol{r}) = 0$$

得到

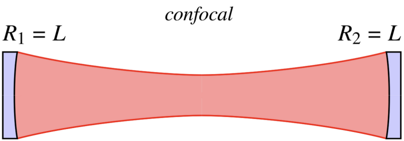
$$\boldsymbol{u}_l(\boldsymbol{r}) = C_1 e^{-i\boldsymbol{k}_l\cdot\boldsymbol{r}} + C_2 e^{i\boldsymbol{k}_l\cdot\boldsymbol{r}}$$

边界条件

$$C_1 + C_2 = 0, \quad C_1 e^{-ik_l L} + C_2 e^{ik_l L} = 0$$

$$\rightarrow \boldsymbol{u}_l(x) = C_1 e^{-ik_l x} - C_1 e^{ik_l x} \quad \text{where} \quad k_l L = n\pi$$

谐振腔内电磁场的量子化



$$\nabla^2 \boldsymbol{u}_l(\boldsymbol{r}) + k_l^2 \boldsymbol{u}_l(\boldsymbol{r}) = 0$$

得到

$$\boldsymbol{u}_l(\boldsymbol{r}) = C_1 e^{-i\boldsymbol{k}_l \cdot \boldsymbol{r}} + C_2 e^{i\boldsymbol{k}_l \cdot \boldsymbol{r}}$$

边界条件

$$C_1 + C_2 = 0, \quad C_1 e^{-ik_l L} + C_2 e^{ik_l L} = 0$$

$$\rightarrow \boldsymbol{u}_l(x) = \sqrt{\frac{2}{L}} \sin(k_l x) \quad \text{where} \quad k_l L = n\pi$$

谐振腔内电磁场的量子化

腔内电磁场的总能量

$$
\begin{aligned}
H_{\mathrm{EM}} &= \frac{1}{2}\int_V \left(\epsilon_0 \boldsymbol{E}^2 + \mu_0 \boldsymbol{H}^2\right) \mathrm{d}V \\
&= \frac{1}{2}\int_V \mathrm{d}V \left[\sum_{m,n} \dot{q}_m \dot{q}_n \boldsymbol{u}_m(\boldsymbol{r}) \cdot \boldsymbol{u}_n(\boldsymbol{r}) + c^2 q_m(t) q_n(t) (\nabla \times \boldsymbol{u}_m) \cdot (\nabla \times \boldsymbol{u}_n)\right]
\end{aligned}
$$

谐振腔内电磁场的量子化

$$\nabla \cdot [(\nabla \times \boldsymbol{u}_m) \times \boldsymbol{u}_n] = (\nabla \times \nabla \times \boldsymbol{u}_m) \cdot \boldsymbol{u}_n - (\nabla \times \boldsymbol{u}_m) \cdot (\nabla \times \boldsymbol{u}_n)$$

得到积分

$$\begin{aligned}
\int (\nabla \times \boldsymbol{u}_m) \cdot (\nabla \times \boldsymbol{u}_n) \, \mathrm{d}V &= \int (\nabla \times \nabla \times \boldsymbol{u}_m) \cdot \boldsymbol{u}_n \, \mathrm{d}V - \int \nabla \cdot [(\nabla \times \boldsymbol{u}_m) \times \boldsymbol{u}_n] \, \mathrm{d}V \\
&= \int (\nabla \times \nabla \times \boldsymbol{u}_m) \cdot \boldsymbol{u}_n \, \mathrm{d}V - \int_{\partial V} [(\nabla \times \boldsymbol{u}_m) \times \boldsymbol{u}_n] \cdot \boldsymbol{n} \, \mathrm{d}\boldsymbol{S} \\
&= \int (\nabla \times \nabla \times \boldsymbol{u}_m) \cdot \boldsymbol{u}_n \, \mathrm{d}V + \int_{\partial V} (\nabla \times \boldsymbol{u}_m) \cdot (\boldsymbol{u}_n \times \boldsymbol{n}) \, \mathrm{d}\boldsymbol{S} \\
&= \int k_m^2 \boldsymbol{u}_m \cdot \boldsymbol{u}_n \, \mathrm{d}V \\
&= k_m^2 \delta_{mn}
\end{aligned}$$

- 高斯散度定理

谐振腔内电磁场的量子化

腔内电磁场的总能量

$$
\begin{aligned}
H_{\mathrm{EM}} &= \frac{1}{2}\int_V \left(\epsilon_0 \boldsymbol{E}^2 + \mu_0 \boldsymbol{H}^2\right) \mathrm{d}V \\
&= \frac{1}{2}\int_V \left[\sum_{m,n} \dot{q}_m \dot{q}_n \boldsymbol{u}_m(\boldsymbol{r}) \boldsymbol{u}_n(\boldsymbol{r}) + c^2 q_m(t) q_n(t) (\nabla \times \boldsymbol{u}_m)(\nabla \times \boldsymbol{u}_n)\right] \mathrm{d}V \\
&= \frac{1}{2}\sum_m \left(\dot{q}_m^2 + c^2 k_m^2 q_m^2\right) \\
&\equiv \sum_m \left(\frac{\dot{q}_m^2}{2} + \frac{1}{2}\omega_m^2 q_m^2\right)
\end{aligned}
$$

- 频率 $\omega_m = c k_m$
- 标准的谐振子模型
- 电磁场是大量无耦合、单位质量的谐振子的集合

谐振腔内电磁场的量子化

对于第 l 个模式

$$H_l = \frac{\dot{q}_l^2}{2} + \frac{1}{2}\omega_l^2 q_l^2 \to \frac{\hat{p}_l^2}{2} + \frac{1}{2}\omega_l^2 \hat{q}_l^2$$

- 定义谐振子的产生和湮灭算符

$$\hat{a}_l = \frac{1}{\sqrt{2\hbar\omega_l}}\left(\omega_l \hat{q}_l + i\hat{p}_l\right), \quad \hat{a}_l^\dagger = \frac{1}{\sqrt{2\hbar\omega_l}}\left(\omega_l \hat{q}_l - i\hat{p}_l\right)$$

- 哈密顿量写为

$$H_{\mathrm{EM}} = \sum_l \hbar\omega\left(a_l^\dagger a_l + \frac{1}{2}\right)$$

谐振腔内电磁场的量子化

在海森堡绘景下

$$i\hbar\frac{\mathrm{d}\hat{a}_l(t)}{\mathrm{d}t} = \left[\hat{a}_l(t), \hat{H}\right] = \hbar\omega_l\hat{a}_l(t)$$

得到

$$\hat{a}_l(t) = e^{-i\omega_l t}\hat{a}_l(0)$$

谐振腔内电磁场的量子化

把

$$\hat{q}_l = \sqrt{\frac{\hbar}{2\omega_l}}\left(\hat{a}_l + \hat{a}_l^\dagger\right), \quad \hat{p}_l = \frac{1}{i}\sqrt{\frac{\hbar\omega_l}{2}}\left(\hat{a}_l - \hat{a}_l^\dagger\right)$$

代回电磁场的表达式

$$\hat{\boldsymbol{A}}(\boldsymbol{r},t) = \frac{1}{\sqrt{\epsilon_0}}\sum_l \hat{q}_l \boldsymbol{u}_l(\boldsymbol{r}) = \sum_l \sqrt{\frac{\hbar}{2\omega_l\epsilon_0}}\left[\hat{a}_l(t) + \hat{a}_l^\dagger(t)\right]\boldsymbol{u}_l(\boldsymbol{r})$$

$$\hat{\boldsymbol{E}}(\boldsymbol{r},t) = -\frac{\partial \hat{\boldsymbol{A}}(\boldsymbol{r},t)}{\partial t} = \sum_l i\sqrt{\frac{\hbar\omega_l}{2\epsilon_0}}\left[\hat{a}_l(t) - \hat{a}_l^\dagger(t)\right]\boldsymbol{u}_l(\boldsymbol{r})$$

$$\hat{\boldsymbol{H}}(\boldsymbol{r},t) = \frac{1}{\mu_0}\nabla\times\hat{\boldsymbol{A}}(\boldsymbol{r},t) = \sum_l \frac{1}{\mu_0}\sqrt{\frac{\hbar}{2\omega_l\epsilon_0}}\left[\hat{a}_l(t) + \hat{a}_l^\dagger\right]\nabla\times\boldsymbol{u}_l(\boldsymbol{r})$$

谐振腔内电磁场的量子化

在 Fock 态中，电磁场的期望值

$$\langle n|\hat{\boldsymbol{E}}|n\rangle = 0, \quad \langle n|\hat{\boldsymbol{H}}|n\rangle = 0$$

这有什么物理解释?

谐振腔内电磁场的量子化

在 Fock 态中，电磁场的期望值

$$\langle n|\hat{\boldsymbol{E}}|n\rangle = 0, \quad \langle n|\hat{\boldsymbol{H}}|n\rangle = 0$$

这有什么物理解释?

- 相位不确定性：不确定关系 $\Delta n \Delta\phi \geq \frac{1}{2}$
- 对于 Fock 态，粒子数是确定的，也就是 $\Delta n = 0$，必须要求 $\Delta\phi$ 无限大，因此 ϕ 趋向于均匀分布在 $[0, 2\pi]$ 区间
- 量子涨落：$\Delta E^2 = \langle n|\hat{\boldsymbol{E}}^2|n\rangle - \langle n|\hat{\boldsymbol{E}}|n\rangle^2 \neq 0$
- 真空涨落：$n = 0$ 时，仍然有 $\Delta E^2 \neq 0$

  $\rightarrow$ 卡西米尔效应 (Casimir effect)
- 与经典物理的对比：经典电磁场，单色平面波有明确的振福和相位

谐振腔内电磁场的量子化

基态的能量无穷大

$$\langle 0|H_{\mathrm{EM}}|0\rangle = \sum_l \frac{1}{2}\hbar\omega_l$$

l 有无穷多个，也就是基态能量无穷大，这有什么物理意义?

自由空间电磁场的量子化

$$\nabla^2 \boldsymbol{A}(\boldsymbol{r},t) - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0$$

- 设电磁场

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{1}{\sqrt{\epsilon_0}}\sum q(t)\boldsymbol{u}(\boldsymbol{r})$$

- ∇^2 的本征模式是什么？

$$-\nabla^2 \boldsymbol{u}(\boldsymbol{r}) = \frac{\omega^2}{c^2}\boldsymbol{u}(\boldsymbol{r})$$

- 库仑规范 $\nabla \cdot \boldsymbol{A}(\boldsymbol{r},t) = 0 \to \nabla \cdot \boldsymbol{u}(\boldsymbol{r}) = 0$

自由空间电磁场的量子化

$$\nabla^2 \boldsymbol{A}(\boldsymbol{r}, t) - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0$$

- 本征模式

$$\boldsymbol{u}(\boldsymbol{r}) = \boldsymbol{\varepsilon}(\boldsymbol{k})\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{\sqrt{V}}$$

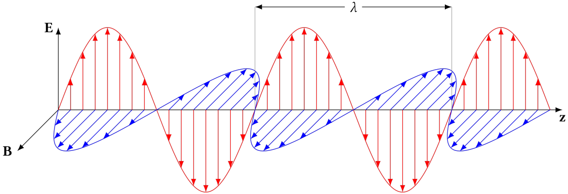
  $\varepsilon(\boldsymbol{k})$ 是极化矢量

- 本征值

$$\begin{aligned}(\boldsymbol{k}\cdot\boldsymbol{k})\boldsymbol{u}(\boldsymbol{r}) &= \frac{\omega^2}{c^2}\boldsymbol{u}(\boldsymbol{r}) \\ \rightarrow \omega^2 &= c^2(\boldsymbol{k}\cdot\boldsymbol{k}) \\ \omega &= kc\end{aligned}$$

自由空间电磁场的量子化

$$\nabla^2 \boldsymbol{A}(\boldsymbol{r}, t) - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0$$



$\nabla \cdot \boldsymbol{u}(\boldsymbol{r}) = 0$ 得到 $\boldsymbol{k} \cdot \boldsymbol{\varepsilon} = 0$

$\rightarrow$ 横波

$\rightarrow$ 一个 $\boldsymbol{k}$, 极化方向有两个自由度

自由空间电磁场的量子化

$$\nabla^2 \boldsymbol{A}(\boldsymbol{r},t) - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0$$

- 电磁场

$$\boldsymbol{A}(\boldsymbol{r},t) = \sum_{\boldsymbol{k}}\sum_{j=1}^{2} \frac{q_j(\boldsymbol{k},t)}{\sqrt{\epsilon_0}}\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{\sqrt{V}}\boldsymbol{\varepsilon}_j(\boldsymbol{k})$$

- $\boldsymbol{A}(\boldsymbol{r},t)$ 是实的，要求

$$q_j(-\boldsymbol{k},t)\boldsymbol{\varepsilon}_j(-\boldsymbol{k}) = q_j^*(\boldsymbol{k},t)\boldsymbol{\varepsilon}_j(\boldsymbol{k})$$

可以取： $$q_j(-\boldsymbol{k},t) = q_j^*(\boldsymbol{k},t), \quad \boldsymbol{\varepsilon}_j(\boldsymbol{k}) = \varepsilon_j(-\boldsymbol{k})$$

- 考虑的是一个边长 L 的正方体，存在周期性边界条件，当 $L \to \infty$ 自然过渡到自由空间

$$e^{ik_x x} = e^{ik_x(x+L)} \to k_x = \frac{2\pi m}{L}, \quad m = 0, \pm1, \pm2, \ldots$$

$$e^{ik_y y} = e^{ik_y(y+L)} \to k_y = \frac{2\pi n}{L}, \quad n = 0, \pm1, \pm2, \ldots$$

$$e^{ik_z z} = e^{ik_z(z+L)} \to k_x = \frac{2\pi l}{L}, \quad l = 0, \pm1, \pm2, \ldots$$

自由空间电磁场的量子化

$$\nabla^2 \boldsymbol{A}(\boldsymbol{r}, t) - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0$$

当 L 趋于无穷，$\Delta k \to 0$，把求和变成积分

$$\sum_k \to \frac{1}{\Delta k^3}\int \mathrm{d}k_x\,\mathrm{d}k_y\,\mathrm{d}k_z \to \left(\frac{L}{2\pi}\right)^3 \int \mathrm{d}\boldsymbol{k}$$

自由空间电磁场的量子化

$$\nabla^2 \boldsymbol{A}(\boldsymbol{r},t) - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0$$

得到

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{V}{(2\pi)^3}\int[\mathrm{d}\boldsymbol{k}]\sum_j \frac{q_j(\boldsymbol{k},t)}{\sqrt{\epsilon_0}}\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{\sqrt{V}}\boldsymbol{\varepsilon}_j(\boldsymbol{k})$$

$$\boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{A}}{\partial t} = -\frac{V}{(2\pi)^3}\int[\mathrm{d}\boldsymbol{k}]\sum_j \frac{\dot{q}_j(\boldsymbol{k},t)}{\sqrt{\epsilon_0}}\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{\sqrt{V}}\boldsymbol{\varepsilon}_j(\boldsymbol{k})$$

$$\boldsymbol{H}(\boldsymbol{r},t) = \frac{\nabla\times\boldsymbol{A}}{\mu_0} = \frac{V}{(2\pi)^3}\int[\mathrm{d}\boldsymbol{k}]\sum_j \frac{q_j(\boldsymbol{k},t)}{\mu_0\sqrt{\epsilon_0}}\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{\sqrt{V}}[i\boldsymbol{k}\times\boldsymbol{\varepsilon}_j(\boldsymbol{k})]$$

自由空间电磁场的量子化

$$
\begin{aligned}
H &= \int \mathrm{d}\boldsymbol{r} \left(\frac{\varepsilon_0}{2} \boldsymbol{E}^2 + \frac{\mu_0}{2} \boldsymbol{H}^2 \right) \\
&= \frac{V}{(2\pi)^3} \int \mathrm{d}\boldsymbol{k} \sum_j \left[\frac{\dot{q}_j^*(\boldsymbol{k},t)\dot{q}_j(\boldsymbol{k},t)}{2} + \frac{1}{2}\omega_k^2 q_j^*(\boldsymbol{k},t) q_j(\boldsymbol{k},t) \right]
\end{aligned}
$$

- 注意与标准谐振子的不同
- $q_j(-\boldsymbol{k},t) = q_j^*(\boldsymbol{k},t)$
- 算符化

$$
\begin{aligned}
\dot{q}_j(\boldsymbol{k},t) \rightarrow p_j(\boldsymbol{k},t) \rightarrow \hat{p}_j(\boldsymbol{k},t) \\
q_j(\boldsymbol{k},t) \rightarrow \hat{q}_j(\boldsymbol{k},t)
\end{aligned}
$$

- 对易关系

$$
\left[\hat{q}_r(\boldsymbol{k},t), \hat{p}_s^\dagger(\boldsymbol{k}',t)\right] = i\hbar\delta_{rs}\delta_{\boldsymbol{k},\boldsymbol{k}'}
$$

or

$$
\left[\hat{q}_r(\boldsymbol{k},t), \hat{p}_s(\boldsymbol{k}',t)\right] = i\hbar\delta_{rs}\delta_{\boldsymbol{k},\boldsymbol{k}'}
$$

自由空间电磁场的量子化

$$
\begin{aligned}
H &= \int \mathrm{d}\boldsymbol{r} \left(\frac{\varepsilon_0}{2} \boldsymbol{E}^2 + \frac{\mu_0}{2} \boldsymbol{H}^2 \right) \\
&= \frac{V}{(2\pi)^3} \int \mathrm{d}\boldsymbol{k} \sum_j \left[\frac{\dot{q}_j^*(\boldsymbol{k},t)\dot{q}_j(\boldsymbol{k},t)}{2} + \frac{1}{2}\omega_k^2 q_j^*(\boldsymbol{k},t) q_j(\boldsymbol{k},t) \right]
\end{aligned}
$$

- 注意与标准谐振子的不同
- $q_j(-\boldsymbol{k},t) = q_j^*(\boldsymbol{k},t)$
- 算符化

$$
\begin{aligned}
\dot{q}_j(\boldsymbol{k},t) \to p_j(\boldsymbol{k},t) &\to \hat{p}_j(\boldsymbol{k},t) \\
q_j(\boldsymbol{k},t) &\to \hat{q}_j(\boldsymbol{k},t)
\end{aligned}
$$

- 对易关系

$$
\left[\hat{q}_r(\boldsymbol{k},t), \hat{p}_s^\dagger(\boldsymbol{k}',t) \right] = i\hbar\delta_{rs}\delta_{\boldsymbol{k},\boldsymbol{k}'} \checkmark
$$

or

$$
\left[\hat{q}_r(\boldsymbol{k},t), \hat{p}_s(\boldsymbol{k}',t) \right] = i\hbar\delta_{rs}\delta_{\boldsymbol{k},\boldsymbol{k}'} \times
$$

自由空间电磁场的量子化

如果 $[\hat{q}_r(\boldsymbol{k},t),\hat{p}_s(\boldsymbol{k}',t)]=i\hbar\delta_{rs}\delta_{\boldsymbol{k},\boldsymbol{k}'}$，那么

$$
\begin{aligned}
&\hat{q}_r(\boldsymbol{k},t)\hat{p}_s(\boldsymbol{k}',t)-\hat{p}_s(\boldsymbol{k}',t)\hat{q}_r(\boldsymbol{k},t)=i\hbar\delta_{rs}\delta_{\boldsymbol{k},\boldsymbol{k}'}\\
\Longrightarrow\ &\hat{p}_s^\dagger(\boldsymbol{k}',t)\hat{q}_r^\dagger(\boldsymbol{k},t)-\hat{q}_r^\dagger(\boldsymbol{k},t)\hat{p}_s^\dagger(\boldsymbol{k}',t)=-i\hbar\delta_{rs}\delta_{\boldsymbol{k},\boldsymbol{k}'}\\
\Longrightarrow\ &\hat{p}_s(-\boldsymbol{k}',t)\hat{q}_r(-\boldsymbol{k},t)-\hat{q}_r(-\boldsymbol{k},t)\hat{p}_s(-\boldsymbol{k}',t)=-i\hbar\delta_{rs}\delta_{\boldsymbol{k},\boldsymbol{k}'}\\
\Longrightarrow\ &\hat{p}_s(-\boldsymbol{k}',t)\hat{q}_r(-\boldsymbol{k},t)-\hat{q}_r(-\boldsymbol{k},t)\hat{p}_s(-\boldsymbol{k}',t)=-i\hbar\delta_{rs}\delta_{-\boldsymbol{k},-\boldsymbol{k}'}\\
\Longrightarrow\ &\hat{p}_s(\boldsymbol{k}',t)\hat{q}_r(\boldsymbol{k},t)-\hat{q}_r(\boldsymbol{k},t)\hat{p}_s(\boldsymbol{k}',t)=-i\hbar\delta_{rs}\delta_{\boldsymbol{k},\boldsymbol{k}'}\leftarrow\quad \text{与第一行矛盾}
\end{aligned}
$$

如果 $[\hat{q}_r(\boldsymbol{k},t),\hat{p}_s(\boldsymbol{k}',t)]=i\hbar\delta_{rs}\delta_{\boldsymbol{k},\boldsymbol{k}'}$，那么

$$\hat{q}_r(\boldsymbol{k},t)\hat{p}_s(\boldsymbol{k}',t)-\hat{p}_s(\boldsymbol{k}',t)\hat{q}_r(\boldsymbol{k},t)=i\hbar\delta_{rs}\delta_{\boldsymbol{k},\boldsymbol{k}'}$$

$$\Longrightarrow \hat{p}_s^\dagger(\boldsymbol{k}',t)\hat{q}_r^\dagger(\boldsymbol{k},t)-\hat{q}_r^\dagger(\boldsymbol{k},t)\hat{p}_s^\dagger(\boldsymbol{k}',t)=-i\hbar\delta_{rs}\delta_{\boldsymbol{k},\boldsymbol{k}'}$$

$$\Longrightarrow \hat{p}_s(-\boldsymbol{k}',t)\hat{q}_r(-\boldsymbol{k},t)-\hat{q}_r(-\boldsymbol{k},t)\hat{p}_s(-\boldsymbol{k}',t)=-i\hbar\delta_{rs}\delta_{\boldsymbol{k},\boldsymbol{k}'}$$

$$\Longrightarrow \hat{p}_s(-\boldsymbol{k}',t)\hat{q}_r(-\boldsymbol{k},t)-\hat{q}_r(-\boldsymbol{k},t)\hat{p}_s(-\boldsymbol{k}',t)=-i\hbar\delta_{rs}\delta_{-\boldsymbol{k},-\boldsymbol{k}'}$$

$$\Longrightarrow \hat{p}_s(\boldsymbol{k}',t)\hat{q}_r(\boldsymbol{k},t)-\hat{q}_r(\boldsymbol{k},t)\hat{p}_s(\boldsymbol{k}',t)=-i\hbar\delta_{rs}\delta_{\boldsymbol{k},\boldsymbol{k}'}\leftarrow$$ 与第一行矛盾

所以，只能是

$$\left[\hat{q}_r(\boldsymbol{k},t),\hat{p}_s^\dagger(\boldsymbol{k}',t)\right]=i\hbar\delta_{rs}\delta_{\boldsymbol{k},\boldsymbol{k}'}$$

自由空间电磁场的量子化

定义算符

$$\hat{a}_j(\boldsymbol{k},t) = \frac{1}{\sqrt{2\hbar\omega_k}}\left[\omega_k \hat{q}_j(\boldsymbol{k},t) + i\hat{p}_j(\boldsymbol{k},t)\right]$$

$$\hat{a}_j^\dagger(\boldsymbol{k},t) = \frac{1}{\sqrt{2\hbar\omega_k}}\left[\omega_k \hat{q}_j^\dagger(\boldsymbol{k},t) - i\hat{p}_j^\dagger(\boldsymbol{k},t)\right]$$

对易关系

$$\left[\hat{a}_r(\boldsymbol{k},t), \hat{a}_s^\dagger(\boldsymbol{k}',t)\right] = \delta_{rs}\delta_{\boldsymbol{k},\boldsymbol{k}'}$$

$$\left[\hat{a}_r(\boldsymbol{k},t), \hat{a}_s(\boldsymbol{k}',t)\right] = 0$$

把 $\hat{q}_j$ 和 $\hat{p}_j$ 代回哈密顿量

$$\hat{H} = \frac{V}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k} \sum_j \hbar\omega_k \left[\hat{a}_j^\dagger(\boldsymbol{k},t)\hat{a}_j(\boldsymbol{k},t) + \frac{1}{2}\right]$$

自由空间电磁场的量子化

场算符可以写为

$$\hat{\boldsymbol{A}}(\boldsymbol{r},t)=\frac{V}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\sum_j\sqrt{\frac{\hbar}{2\omega_k\epsilon_0}}\left[\hat{a}_j(\boldsymbol{k},t)+\hat{a}_j^\dagger(-\boldsymbol{k},t)\right]\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{\sqrt{V}}\varepsilon_j(\boldsymbol{k})$$

$$\hat{\boldsymbol{E}}(\boldsymbol{r},t)=-\frac{\partial\hat{\boldsymbol{A}}}{\partial t}=\frac{V}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\sum_j i\sqrt{\frac{\hbar\omega_k}{2\epsilon_0}}\left[\hat{a}_j(\boldsymbol{k},t)-\hat{a}_j^\dagger(-\boldsymbol{k},t)\right]\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{\sqrt{V}}\varepsilon_j(\boldsymbol{k})$$

$$\hat{\boldsymbol{H}}(\boldsymbol{r},t)=\frac{\nabla\times\hat{\boldsymbol{A}}}{\mu_0}=\frac{V}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\sum_j\frac{1}{\mu_0}\sqrt{\frac{\hbar}{2\omega_k\epsilon_0}}\left[\hat{a}_j(\boldsymbol{k},t)+\hat{a}_j^\dagger(-\boldsymbol{k},t)\right]\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{\sqrt{V}}[i\boldsymbol{k}\times\varepsilon_j(\boldsymbol{k})]$$

自由空间电磁场的量子化

海森堡方程

$$i\hbar \frac{\partial \hat{a}_j(\boldsymbol{r},t)}{\partial t} = \left[\hat{a}_j(\boldsymbol{k},t), \hat{H}\right] = \hbar\omega_k \hat{a}_j(\boldsymbol{k},t)$$

得到

$$\hat{a}_j(\boldsymbol{k},t) = e^{-i\omega_k t}\hat{a}_j(\boldsymbol{k},0) = e^{-i\omega_k t}\hat{a}_j(\boldsymbol{k})$$

自由空间电磁场的量子化

场算符可以写为

$$\hat{\boldsymbol{A}}(\boldsymbol{r},t)=\frac{V}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\sum_j\sqrt{\frac{\hbar}{2\omega_k\epsilon_0}}\left[\hat{a}_j(\boldsymbol{k})e^{-i\omega_k t}+\hat{a}_j^\dagger(-\boldsymbol{k})e^{i\omega_{-k}t}\right]\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{\sqrt{V}}\varepsilon_j(\boldsymbol{k})$$

$$\hat{\boldsymbol{E}}(\boldsymbol{r},t)=\frac{V}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\sum_j i\sqrt{\frac{\hbar\omega_k}{2\epsilon_0}}\left[\hat{a}_j(\boldsymbol{k})e^{-i\omega_k t}-\hat{a}_j^\dagger(-\boldsymbol{k})e^{i\omega_{-k}t}\right]\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{\sqrt{V}}\varepsilon_j(\boldsymbol{k})$$

$$\hat{\boldsymbol{H}}(\boldsymbol{r},t)=\frac{V}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\sum_j \frac{1}{\mu_0}\sqrt{\frac{\hbar}{2\omega_k\epsilon_0}}\left[\hat{a}_j(\boldsymbol{k})e^{-i\omega_k t}+\hat{a}_j^\dagger(-\boldsymbol{k})e^{i\omega_{-k}t}\right]\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{\sqrt{V}}[i\boldsymbol{k}\times\varepsilon_j(\boldsymbol{k})]$$

自由空间电磁场的量子化

场算符可以写为

$$\hat{\boldsymbol{A}}(\boldsymbol{r},t)=\frac{V}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\sum_j\sqrt{\frac{\hbar}{2\omega_k\epsilon_0}}\left[\hat{a}_j(\boldsymbol{k})e^{-i\omega_k t+i\boldsymbol{k}\cdot\boldsymbol{r}}\frac{\varepsilon_j(\boldsymbol{k})}{\sqrt{V}}+\mathrm{H.\,c.}\right]$$

$$\hat{\boldsymbol{E}}(\boldsymbol{r},t)=\frac{V}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\sum_j\sqrt{\frac{\hbar\omega_k}{2\epsilon_0}}\left[i\hat{a}_j(\boldsymbol{k})e^{-i\omega_k t+i\boldsymbol{k}\cdot\boldsymbol{r}}\frac{\varepsilon_j(\boldsymbol{k})}{\sqrt{V}}+\mathrm{H.\,c.}\right]$$

$$\hat{\boldsymbol{H}}(\boldsymbol{r},t)=\frac{V}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\sum_j\frac{1}{\mu_0}\sqrt{\frac{\hbar}{2\omega_k\epsilon_0}}\left[\hat{a}_j(\boldsymbol{k})e^{-i\omega_k t+i\boldsymbol{k}\cdot\boldsymbol{r}}\frac{i\boldsymbol{k}\times\varepsilon_j(\boldsymbol{k})}{\sqrt{V}}+\mathrm{H.\,c.}\right]$$

动量算符

$$\begin{aligned}\hat{\boldsymbol{P}} =&\epsilon_0\mu_0 \int \mathrm{d}\boldsymbol{r}\, \hat{\boldsymbol{E}}(\boldsymbol{r},t)\times \hat{\boldsymbol{H}}(\boldsymbol{r},t)\\ =&\frac{V^2}{(2\pi)^6}\frac{\hbar}{2}\int \mathrm{d}\boldsymbol{r}\int \mathrm{d}\boldsymbol{k}\int \mathrm{d}\boldsymbol{k}'\sum_{j,r}\left[i\hat{a}_j(\boldsymbol{k})e^{-i\omega_k t+i\boldsymbol{k}\cdot\boldsymbol{r}}\frac{\varepsilon_j(\boldsymbol{k})}{\sqrt{V}}+\mathrm{H.c.}\right]\\ &\times\left[\hat{a}_r(\boldsymbol{k}')e^{-i\omega_{k'}t+i\boldsymbol{k}'\cdot\boldsymbol{r}}\frac{i\boldsymbol{k}'\times\varepsilon_r(\boldsymbol{k}')}{\sqrt{V}}+\mathrm{H.c.}\right]\end{aligned}$$

利用公式 $\frac{1}{(2\pi)^3}\int \mathrm{d}\boldsymbol{r}\, e^{i(\boldsymbol{k}-\boldsymbol{k}')\cdot\boldsymbol{r}} = \delta(\boldsymbol{k}-\boldsymbol{k}')$

$$\begin{aligned}\hat{\boldsymbol{P}} &= \frac{V}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\sum_j \hbar\boldsymbol{k}\left[\hat{a}_j^\dagger(\boldsymbol{k})\hat{a}_j(\boldsymbol{k})+\frac{1}{2}\right]\\ &= \frac{V}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\sum_j \hbar\boldsymbol{k}\hat{a}_j^\dagger(\boldsymbol{k})\hat{a}_j(\boldsymbol{k})\end{aligned}$$

- 光子数的本征态也是动量的本征态
- 一个光子的动量为 $\hbar\boldsymbol{k}$

自由空间电磁场的量子化

角动量算符

$$
\begin{aligned}
\hat{\boldsymbol{J}}(t) &= \int \mathrm{d}\boldsymbol{r}\, \boldsymbol{r} \times \epsilon_0 \mu_0 \left[\hat{\boldsymbol{E}}(\boldsymbol{r},t) \times \hat{\boldsymbol{H}}(\boldsymbol{r},t)\right] \\
&= \epsilon_0 \int \mathrm{d}\boldsymbol{r}\, \boldsymbol{r} \times \left[\hat{\boldsymbol{E}}(\boldsymbol{r},t) \times (\nabla \times \hat{\boldsymbol{A}}(\boldsymbol{r},t))\right]
\end{aligned}
$$

角动量算符

$$\hat{\boldsymbol{J}}(t) = \int \mathrm{d}\boldsymbol{r}\, \boldsymbol{r} \times \epsilon_0\mu_0 \left[\hat{\boldsymbol{E}}(\boldsymbol{r},t) \times \hat{\boldsymbol{H}}(\boldsymbol{r},t)\right]$$
$$= \epsilon_0 \int \mathrm{d}\boldsymbol{r}\, \boldsymbol{r} \times \left[\hat{\boldsymbol{E}}(\boldsymbol{r},t) \times (\nabla \times \hat{\boldsymbol{A}}(\boldsymbol{r},t))\right]$$

- 爱因斯坦求和约定推导上面的被积函数
- 利用 $\varepsilon_{klm}\varepsilon_{mnp} = \delta_{kn}\delta_{lp} - \delta_{kp}\delta_{ln}$，得到

$$\varepsilon_{ijk}r_j\varepsilon_{klm}E_l\varepsilon_{mnp}\partial_n A_p = \varepsilon_{ijk}r_jE_l\partial_k A_l - \varepsilon_{ijk}r_jE_l\partial_l A_k$$

- 上面第二项的积分使用分部积分，由于 $\partial_l E_l = 0$ 和 $\partial_l r_j = \delta_{lj}$，被积函数可以写为

$$\varepsilon_{ijk}r_jE_l\partial_k A_l + \varepsilon_{ijk}E_lA_k = \hat{\boldsymbol{E}}_a\left(\boldsymbol{r} \times \nabla\right)\hat{\boldsymbol{A}}_a + \hat{\boldsymbol{E}} \times \hat{\boldsymbol{A}}$$

自由空间电磁场的量子化

角动量算符

$$
\begin{aligned}
\hat{\boldsymbol{J}}(t) &= \int \mathrm{d}\boldsymbol{r}\, \boldsymbol{r} \times \epsilon_0 \mu_0 \left[\hat{\boldsymbol{E}}(\boldsymbol{r}, t) \times \hat{\boldsymbol{H}}(\boldsymbol{r}, t) \right] \\
&= \epsilon_0 \int \mathrm{d}\boldsymbol{r}\, \boldsymbol{r} \times \left[\hat{\boldsymbol{E}}(\boldsymbol{r}, t) \times (\nabla \times \hat{\boldsymbol{A}}(\boldsymbol{r}, t)) \right] \\
&= \epsilon_0 \int \mathrm{d}\boldsymbol{r}\, (\hat{\boldsymbol{E}} \times \hat{\boldsymbol{A}}) + \epsilon_0 \int \mathrm{d}\boldsymbol{r}\, \hat{\boldsymbol{E}}_a \, (\boldsymbol{r} \times \nabla)\, \hat{\boldsymbol{A}}_a
\end{aligned}
$$

自由空间电磁场的量子化

角动量算符

$$\hat{\boldsymbol{J}}(t) = \epsilon_0 \int \mathrm{d}\boldsymbol{r}\,(\hat{\boldsymbol{E}} \times \hat{\boldsymbol{A}}) + \epsilon_0 \int \mathrm{d}\boldsymbol{r}\,\hat{\boldsymbol{E}}_a\,(\boldsymbol{r} \times \nabla)\,\hat{\boldsymbol{A}}_a$$

- 第一项为电磁波的自旋角动量
- 第二项为电磁波的轨道角动量
- 平面波下: 比如场往 z 方向传播 $\boldsymbol{E} \sim e^{\pm i\boldsymbol{k}\cdot\boldsymbol{z}}$, $\boldsymbol{A} \sim e^{\pm i\boldsymbol{k}\cdot\boldsymbol{z}} \Longrightarrow$ 第二项为零

$$\int e^{\pm i\boldsymbol{k}\cdot\boldsymbol{z}}(\boldsymbol{r} \times k_z\vec{\mathbf{z}})e^{\pm i\boldsymbol{k}\cdot\boldsymbol{z}}\,\mathrm{d}\boldsymbol{r} \to 0$$

自由空间电磁场的量子化

自旋角动量

$$\begin{aligned}\hat{\boldsymbol{S}}(t) =&\epsilon_0 \int \mathrm{d}\boldsymbol{r}\, \hat{\boldsymbol{E}}(\boldsymbol{r},t) \times \hat{\boldsymbol{A}}(\boldsymbol{r},t) \\ =&\frac{V^2}{(2\pi)^6} \int \mathrm{d}\boldsymbol{k}\, \mathrm{d}\boldsymbol{k}' \sum_{r,s} \frac{1}{2} \left[i\hat{a}_r(\boldsymbol{k}) e^{-i\omega_k t + i\boldsymbol{k}\cdot\boldsymbol{r}} \frac{\varepsilon_r(\boldsymbol{k})}{\sqrt{V}} + \mathrm{H.\,c.} \right] \\ &\times \left[\hat{a}_s(\boldsymbol{k}') e^{-i\omega_{k'} t + i\boldsymbol{k}'\cdot\boldsymbol{r}} \frac{\varepsilon_s(\boldsymbol{k}')}{\sqrt{V}} + \mathrm{H.\,c.} \right]\end{aligned}$$

- 极化矢量满足

$$\varepsilon_j(-\boldsymbol{k}) = \varepsilon_j(\boldsymbol{k}), \quad \varepsilon_1(\boldsymbol{k}) \times \varepsilon_2(\boldsymbol{k}) = \widehat{\mathbf{k}},$$

- 叉乘的存在只能是 $r \neq s$

$$\hat{\boldsymbol{S}}(t) = \frac{V}{(2\pi)^3} \int \mathrm{d}\boldsymbol{k}\, i\hbar \widehat{\mathbf{k}} \left[\hat{a}_2^\dagger(\boldsymbol{k})\hat{a}_1(\boldsymbol{k}) - \hat{a}_1^\dagger(\boldsymbol{k})\hat{a}_2(\boldsymbol{k}) \right]$$

自旋角动量

$$\hat{\boldsymbol{S}}(t) = \frac{V}{(2\pi)^3} \int \mathrm{d}\boldsymbol{k}\, i\hbar \widehat{\mathbf{k}} \left[\hat{a}_2^\dagger(\boldsymbol{k})\hat{a}_1(\boldsymbol{k}) - \hat{a}_1^\dagger(\boldsymbol{k})\hat{a}_2(\boldsymbol{k})\right]$$

- $\hat{S}$ 的的本征态是什么？

自由空间电磁场的量子化

自旋角动量

$$\hat{\boldsymbol{S}}(t) = \frac{V}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\, i\hbar\widehat{\mathbf{k}}\left[\hat{a}_2^\dagger(\boldsymbol{k})\hat{a}_1(\boldsymbol{k}) - \hat{a}_1^\dagger(\boldsymbol{k})\hat{a}_2(\boldsymbol{k})\right]$$

- $\hat{S}$ 的的本征态是什么？
- 做线性变换

$$\hat{a}_R = \frac{\hat{a}_1(\boldsymbol{k}) - i\hat{a}_2(\boldsymbol{k})}{\sqrt{2}}, \quad \hat{a}_L = \frac{\hat{a}_1(\boldsymbol{k}) + i\hat{a}_2(\boldsymbol{k})}{\sqrt{2}}$$

满足对易关系

$$\left[\hat{a}_L(\boldsymbol{k}), \hat{a}_L(\boldsymbol{k}')\right] = \delta_{\boldsymbol{k},\boldsymbol{k}'}, \quad \left[\hat{a}_L(\boldsymbol{k}), \hat{a}_R(\boldsymbol{k}')\right] = 0$$

自旋角动量

$$\hat{\boldsymbol{S}} = \frac{V}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\,\hbar\widehat{\mathbf{k}}\left[\hat{a}_R^\dagger(\boldsymbol{k})\hat{a}_R - \hat{a}_L(\boldsymbol{k})\hat{a}_L(\boldsymbol{k})\right]$$

- $\hat{a}_{R/L}$ 产生一个光子：处在垂直极化的两个状态的叠加态
- 右旋极化的光子态

$$|1\rangle_{\boldsymbol{k},R} = \hat{a}_R^\dagger(\boldsymbol{k})\,|0\rangle = \frac{\hat{a}_1^\dagger(\boldsymbol{k}) + i\hat{a}_2^\dagger(\boldsymbol{k})}{\sqrt{2}}\,|0\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle_{\boldsymbol{k},1} + i\,|1\rangle_{\boldsymbol{k},2}\right)$$

- 左旋极化的光子态

$$|1\rangle_{\boldsymbol{k},L} = \hat{a}_L^\dagger(\boldsymbol{k})\,|0\rangle = \frac{\hat{a}_1^\dagger(\boldsymbol{k}) - i\hat{a}_2^\dagger(\boldsymbol{k})}{\sqrt{2}}\,|0\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle_{\boldsymbol{k},1} - i\,|1\rangle_{\boldsymbol{k},2}\right)$$

自由空间电磁场的量子化

自旋角动量

$$\hat{\boldsymbol{S}} = \frac{V}{(2\pi)^3} \int \mathrm{d}\boldsymbol{k}\, \hbar\widehat{\mathbf{k}} \left[\hat{a}_R^\dagger(\boldsymbol{k})\hat{a}_R - \hat{a}_L(\boldsymbol{k})\hat{a}_L(\boldsymbol{k})\right]$$

- **本征值**

$$\hat{\boldsymbol{S}} \left|1\right\rangle_{\boldsymbol{k},R} = +\hbar\widehat{\mathbf{k}} \left|1\right\rangle_{\boldsymbol{k},R}$$
$$\hat{\boldsymbol{S}} \left|1\right\rangle_{\boldsymbol{k},L} = -\hbar\widehat{\mathbf{k}} \left|1\right\rangle_{\boldsymbol{k},L}$$

- **自旋方向平行于 $\boldsymbol{k}$ 的方向**

第一章

量子光学简介

量子力学公设

谐振子量子化

电磁场及其量子化

热平衡光场

量子相位算符

总结

热平衡光场

- 系统温度 T，粒子数 $N \to$ 正则系综
- 配分函数

$$
\begin{aligned}
Z &= \mathrm{tr}\left(e^{-\beta\hat{H}}\right) \\
&= \mathrm{tr}\left(\sum_n |n\rangle\langle n|\, e^{-\beta\hat{H}}\right) \\
&= \sum_n \langle n|e^{-\beta\hat{H}}|n\rangle = \sum_n e^{-\beta\hbar\omega(n+\frac{1}{2})} \\
&= \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}}
\end{aligned}
$$

- 第 n 个 Fock 态的概率

$$
P_n = \frac{1}{Z}e^{-\beta E_n}
$$

热平衡光场

- 系统温度 T，粒子数 $N \rightarrow$ 正则系综
- 配分函数

$$
\begin{aligned}
Z &= \mathrm{tr}\left(e^{-\beta\hat{H}}\right) \\
&= \mathrm{tr}\left(\sum_n |n\rangle\langle n|\, e^{-\beta\hat{H}}\right) \\
&= \sum_n \langle n|e^{-\beta\hat{H}}|n\rangle = \sum_n e^{-\beta\hbar\omega(n+\frac{1}{2})} \\
&= \frac{e^{-\beta\hbar\omega/2}}{1-e^{-\beta\hbar\omega}}
\end{aligned}
$$

- 第 n 个 Fock 态的概率

$$
P_n = \frac{1}{Z} e^{-\beta E_n}
$$

- 密度矩阵

$$
\rho = \sum_n P_n\, |n\rangle\langle n| = \sum_{n=0}^{\infty} \frac{1-e^{-\beta\hbar\omega}}{e^{n\beta\hbar\omega}}\, |n\rangle\langle n|
$$

热平衡光场

平均粒子数

$$\bar{n} = \mathrm{tr}(\hat{a}\rho) = (1 - e^{-\beta\hbar\omega}) \sum_{n=0}^{\infty} e^{-n\beta\hbar\omega} n$$

- 设 $\lambda = \beta\hbar\omega$，有 $g(\lambda) = \sum_{n=0}^{\infty} e^{-n\lambda} = \frac{1}{1-e^{-\lambda}}$

$$\bar{n} = (1 - e^{-\lambda}) g'(\lambda) = \frac{1}{e^{\lambda} - 1} = \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1}$$

- 玻色-爱因斯坦分布
- 光子能量越高，平均粒子数越小
- 室温下，波长 $\lambda \in (10, 100)\mu$m，平均粒子数约为 1

密度矩阵

$$\rho = \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} |n\rangle\langle n|$$

热库系统与主方程

$$\langle n^2 \rangle = \mathrm{tr}\left(\hat{n}^2 \rho\right) = 2\bar{n}^2 + \bar{n}$$

粒子数涨落

$$\Delta n = \sqrt{\langle n^2 \rangle - \langle n \rangle^2} = \bar{n}^2 + \bar{n}$$

- 当 $\bar{n} \gg 1, \to \Delta n \approx \bar{n} + 1/2$
- 当 $\bar{n} \ll 1, \to \Delta n \approx \sqrt{\bar{n}}$

第一章

量子光学简介

量子力学公设

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电磁场及其量子化

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量子相位算符

总结

量子相位算符

任何一个复矩阵 A，可以分解为

$$A = hU, \qquad \text{where} \quad h = \sqrt{AA^\dagger}$$

- U 为幺正矩阵
- 产生和湮灭算符

$$\hat{a}^\dagger = e^{-i\hat{\phi}}\sqrt{\hat{a}\hat{a}^\dagger}, \quad \hat{a} = \sqrt{\hat{a}\hat{a}^\dagger}e^{i\hat{\phi}}$$

- 采用 Susskind 和 Glogower 的定义

$$e^{i\hat{\phi}} = \frac{1}{\sqrt{\hat{a}\hat{a}^\dagger}}\hat{a}, \quad e^{-i\hat{\phi}} = \hat{a}^\dagger\frac{1}{\sqrt{\hat{a}\hat{a}^\dagger}}\hat{a}$$

量子相位算符

$$e^{i\hat{\phi}}|n\rangle = \frac{1}{\sqrt{\hat{a}\hat{a}^\dagger}}\hat{a}|n\rangle = \begin{cases} 0, & n=0 \\ |n-1\rangle, & n\neq 0 \end{cases}$$

$$e^{-i\hat{\phi}}|n\rangle = \hat{a}^\dagger\frac{1}{\sqrt{\hat{a}\hat{a}^\dagger}}|n\rangle = |n+1\rangle$$

- 在 Fock 态表象中：

$$e^{i\hat{\phi}} = \sum_{n=0}^{\infty}|n\rangle\langle n+1|, \quad e^{-i\hat{\phi}} = \sum_{n=0}^{\infty}|n+1\rangle\langle n|$$

- 可以验证：$e^{i\hat{\phi}}e^{-i\hat{\phi}} = I$, $e^{-i\hat{\phi}}e^{i\hat{\phi}} = I - |0\rangle\langle 0|$, $\left[e^{i\hat{\phi}}, e^{-i\hat{\phi}}\right] = |0\rangle\langle 0|$
- 所以，$e^{i\hat{\phi}}$ 不是一个幺正算符 $\rightarrow$ $\hat{\phi}$ 不是厄米算符
- 这与 Fock 态空间的限制密切相关：$n \geq 0$

量子相位算符

引入一对厄米算子

$$\cos\hat{\phi} = \frac{e^{i\hat{\phi}} + e^{-i\hat{\phi}}}{2}, \quad \sin\hat{\phi} = \frac{e^{i\hat{\phi}} - e^{-i\hat{\phi}}}{2i}$$

满足关系

$$[\cos\hat{\phi}, \sin\hat{\phi}] = \frac{i}{2}\left|0\right\rangle\!\left\langle 0\right|$$

$$\cos^2\hat{\phi} + \sin^2\hat{\phi} = I - \frac{1}{2}\left|0\right\rangle\!\left\langle 0\right|$$

$$[\hat{n}, \cos\hat{\phi}] = -i\sin\hat{\phi}$$

$$[\hat{n}, \sin\hat{\phi}] = i\cos\hat{\phi}$$

量子相位算符

从对易关系 $\left[\hat{n}, e^{i\hat{\phi}}\right] = -e^{i\hat{\phi}}$，可以得到

$$
\begin{aligned}
\hat{n} + 1 &= e^{i\hat{\phi}} \hat{n} e^{-i\hat{\phi}} \\
&\overset{*}{=} \hat{n} + i[\hat{\phi}, \hat{n}] + \frac{i^2}{2!}[\hat{\phi}, [\hat{\phi}, \hat{n}]] + \ldots
\end{aligned}
$$

- 这里强行使用了 Baker-Hausdorf 公式
- 对比左右两边，得到

$$\left[\hat{n}, \hat{\phi}\right] = i$$

- 不确定性关系 $(\Delta n)(\Delta \phi) \overset{*}{\geq} 1/2$

量子相位算符

从对易关系 $\left[\hat{n}, e^{i\hat{\phi}}\right] = -e^{i\hat{\phi}}$，可以得到

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- 这里强行使用了 Baker-Hausdorf 公式
- 对比左右两边，得到

$$\left[\hat{n}, \hat{\phi}\right] = i$$

- 不确定性关系 $(\Delta n)(\Delta \phi) \overset{*}{\geq} 1/2$
- 问题 1：如果 Δn 很小时，由于相位只能是 $\in [0, 2\pi]$，所以 $\Delta \phi$ 不可能无限大
- 问题 2：在 Fock 态上

$$(m - n) \langle m|\hat{\phi}|n\rangle \overset{*}{=} i\delta_{mn}$$

若 $m = n$，则得到 $0 = i$

- $e^{i\hat{\phi}}$ 不是可逆算符，BCH 公式的结果不成立

量子相位算符

- 无穷维变成有限的
- 连续的变成离散的

在 $[0, 2\pi]$ 之间取 M 个离散点，因此相位的空间就是 $M+1$，定义参考相位态

$$|\theta_0\rangle = \frac{1}{\sqrt{M+1}} \sum_{n=0}^{M} e^{in\theta_0} |n\rangle$$

构造相位空间的正交基，$\theta_k = \theta_0 + \frac{2k\pi}{M+1}$, $k = 0, 1, \ldots, M$:

$$|\theta_k\rangle = \frac{1}{\sqrt{M+1}} \sum_{n=0}^{M} e^{in\theta_k} |n\rangle$$

量子相位算符

- **正交性**

$$\langle\theta_k|\theta_l\rangle = \frac{1}{M+1}\sum_{m,n}\langle m|e^{-im\theta_k}e^{in\theta_l}|n\rangle$$

- **完备性**

$$\begin{aligned}\sum_{k=0}^{M}|\theta_k\rangle\langle\theta_k| &= \sum_{k=0}^{M}\frac{1}{M+1}\sum_{m,n=0}^{M} e^{in\theta_k}\,|n\rangle\,\langle m|\,e^{-im\theta_k} \\ &= \frac{1}{M+1}\sum_{k}\sum_{n,m} e^{i(n-m)\frac{2k\pi}{M+1}}\,|n\rangle\langle m| \\ &= \sum_{m}|m\rangle\langle m| \\ &= I_{M+1}\end{aligned}$$

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量子相位算符

Fock 态可以用相位空间基矢表示

$$
\begin{aligned}
|n\rangle &= \sum_{k=0}^{M} |\theta_k\rangle \langle\theta_k|n\rangle \\
&= \sum_{k} |\theta_k\rangle \frac{1}{\sqrt{M+1}} \sum_{m} \langle m|e^{-im\theta_k}|n\rangle \\
&= \frac{1}{\sqrt{M+1}} \sum_{k=0}^{M} e^{-in\theta_k} |\theta_k\rangle
\end{aligned}
$$

指数相位算符

$$
e^{i\hat{\phi}} = \sum_{k} e^{i\theta_k} |\theta_k\rangle\langle\theta_k|
$$

由正交性，可以直接得到

$$
e^{i\hat{\phi}} |\theta_k\rangle = e^{i\theta_k} |\theta_k\rangle , \quad \hat{\phi} = \sum_{k} \theta_k |\theta_k\rangle\langle\theta_k|
$$

量子相位算符

$$e^{i\hat{\phi}}|n\rangle = \sum_k e^{i\theta_k}|\theta_k\rangle\langle\theta_k|\frac{1}{\sqrt{M+1}}\sum_{k'} e^{-in\theta_{k'}}|\theta_{k'}\rangle$$
$$= \frac{1}{\sqrt{M+1}}\sum_k e^{-i(n-1)\theta_k}|\theta_k\rangle$$

- 若 $n>0$，则有 $e^{i\hat{\phi}}|n\rangle = |n-1\rangle$
- 若 $n=0$，则

$$e^{i\hat{\phi}}|0\rangle = \frac{1}{\sqrt{M+1}}\sum_k e^{-i(-1)\theta_k}|\theta_k\rangle$$
$$= \frac{1}{\sqrt{M+1}}\sum_k e^{-i(-\theta_0-\frac{2k\pi}{M+1}+2\pi)}|\theta_k\rangle$$
$$= \frac{1}{\sqrt{M+1}}\sum_k e^{-i(-\theta_0+\frac{2Mk\pi}{M+1})}|\theta_k\rangle$$
$$= \frac{1}{\sqrt{M+1}}\sum_k e^{-iM(\theta_0+\frac{2k\pi}{M+1})+i(M+1)\theta_0}|\theta_k\rangle$$
$$= e^{i(M+1)\theta_0}|M\rangle$$

量子相位算符

同理，可以得到

$$e^{-i\hat{\phi}}|n\rangle = \begin{cases} |n+1\rangle, & n < M \\ e^{-i(M+1)\theta_0}, & n = M \end{cases}$$

在 Fock 态表象中

$$e^{i\hat{\phi}} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ e^{i(M+1)\theta_0} & 0 & 0 & \dots & 0 \end{pmatrix}, \quad e^{-i\hat{\phi}} = \begin{pmatrix} 0 & 0 & 0 & \dots & e^{-i(M+1)\theta_0} \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

容易得到

$$e^{-i\hat{\phi}}e^{i\hat{\phi}} = e^{i\hat{\phi}}e^{-i\hat{\phi}} = 1$$

- 这里的 $e^{\pm i\hat{\phi}}$ 是可逆幺正算符

量子相位算符

任意态 $|\psi\rangle = \sum_n c_n |n\rangle$ 的相位分布函数

$$W(\theta_k) = |\langle\theta_k|\psi\rangle|^2 = \frac{1}{M+1}\left|\sum_n c_n e^{-in\theta_k}\right|$$

- 当 $M\to\infty$ 时，求和变成积分

$$\lim_{M\to\infty}\sum_{k=0}^{M} = \frac{1}{\Delta\theta}\int \mathrm{d}\theta = \frac{M+1}{2\pi}\int \mathrm{d}\theta$$

- 分布函数：$\theta\to\theta+\mathrm{d}\theta$ 的概率为

$$W(\theta)\mathrm{d}\theta = \frac{1}{2\pi}\left|\sum_n c_n e^{-in\theta}\right|^2 \mathrm{d}\theta$$

量子相位算符

例子：Fock 态 $|N\rangle$

$$W(\theta) = \frac{1}{2\pi}$$

- 粒子数本征态的相位随机均匀分布

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Summary

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- 量子力学公设
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- 薛定谔方程与表象
- 谐振子量子化
- 电磁场量子化、光场动量和自旋
- 热平衡态
- 量子相位算符