

# 量子光学

## 第二章 量子相干态

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# 课程大纲

量子光学是研究光场的量子性质，及其与原子、分子等物质相互作用的物理学分支，核心内容包括光子的产生、操控与探测以及光-物质的量子相干与纠缠现象。

第一章 辐射场及其量子化 (2 次课)

第二章 量子相干态 (3 次课)

第三章 光场相干性及其干涉 (3 次课)

第四章 光场与原子相互作用 (4 次课)

第五章 热库系统与主方程 (2 次课)

第六章 激光冷却技术 (2 次课)

第七章 光学腔和光力耦合系统 (2 次课)

## 参考资料

- 《量子光学》，郭光灿，周祥发，科学出版社
- Quantum Optics, Marlan O. Scully, M. Suhail Zubairy, Cambridge University Press
- Lecture of Professor Farhan Rana
- 维基百科 (wikipedia.org)

## 第二章

相干态的性质

压缩相干态

光场的 Wigner 特征函数

自旋相干态/原子相干态

总结

## 第二章

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总结

光腔中电流和电磁场的耦合

$$H = \hbar\omega_c \hat{a}^\dagger \hat{a} - \int d\mathbf{r} \mathbf{j}(\mathbf{r}, t) \cdot \hat{\mathbf{A}}(\mathbf{r})$$

- 电流  $\mathbf{j}(\mathbf{r}, t) = S(\mathbf{r}) \cos(\omega_c t - \theta)$
- 电磁场  $\hat{\mathbf{A}}(\mathbf{r}) = \sqrt{\frac{\hbar}{2\omega_c \epsilon_0}} (\hat{a} + \hat{a}^\dagger) \mathbf{u}(\mathbf{r})$
- 只考虑共振情况

$$H_{\text{eff}} = \Omega \hat{a} + \Omega^* \hat{a}^\dagger$$

薛定谔方程

$$\frac{d}{dt} |\psi(t)\rangle = -\frac{i}{\hbar} H_{\text{eff}} |\psi(t)\rangle$$

# 经典电流的辐射

辐射态为

$$\begin{aligned} |\psi(t)\rangle &= \exp\left(-\frac{i}{\hbar} \int_0^t H_{\text{eff}} d\tau\right) |\psi(0)\rangle \\ &= \exp\left(-\frac{i}{\hbar} \Omega t \hat{a} - \frac{i}{\hbar} \Omega^* t \hat{a}^\dagger\right) |\psi(0)\rangle \\ &\equiv \exp\left(\alpha \hat{a}^\dagger - \alpha^* \hat{a}\right) |\psi(0)\rangle \end{aligned}$$

- 设  $\alpha(t) = -\frac{i}{\hbar} \Omega^* t$
- 若初态是真空，这个辐射态是什么态？

$$|\alpha\rangle \equiv e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} |0\rangle$$

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→ 相干态

## 几个公式

- BCH 公式

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A} + \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}] + \frac{1}{12}([\hat{A}, [\hat{A}, \hat{B}]] + [\hat{B}, [\hat{B}, \hat{A}]] + \dots)}$$

如果  $[\hat{A}, \hat{B}]$  是一个常数,

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A} + \hat{B} + \frac{1}{2}[\hat{A}, \hat{B}]}, \quad e^{\hat{A}} e^{\hat{B}} = e^{\hat{B}} e^{\hat{A}} e^{[\hat{A}, \hat{B}]}$$

- BH 公式

$$e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \dots$$

- 矩阵指数的导数

$$\frac{d}{dx} e^{\hat{A}} = e^{\hat{A}} \hat{A}' + \frac{1}{2} e^{\hat{A}} [\hat{A}', \hat{A}] + \frac{1}{3!} [[\hat{A}', \hat{A}], \hat{A}] + \dots$$

## 证明 BH 公式

设  $f(t) = e^{t\hat{A}}\hat{B}e^{-t\hat{A}}$

- 一阶导数

$$\begin{aligned}\frac{d}{dt}f(t) &= \hat{A}e^{t\hat{A}}\hat{B}e^{-t\hat{A}} - e^{t\hat{A}}\hat{B}e^{-t\hat{A}}\hat{A} \\ &= [\hat{A}, f(t)]\end{aligned}$$

- $n$  阶导数

$$\frac{d^n}{dt^n}f(t) = [\hat{A}, [\hat{A}, [\dots, [\hat{A}, f(t)]]]]$$

- $f(t)$  的 Taylor 展开

$$f(t) = f(0) + f'(0)t + \frac{f''(0)}{2!}t^2 + \dots$$

令  $t = 1$ , 即得到 BH 公式

## 证明 BCH 公式

设  $f(x) = e^{xA}e^{xB} = e^{G(x)} \rightarrow$  只需要证明: 当  $x = 1$  时,

$$G(1) = A + B + \frac{1}{2}[A, B] + \frac{1}{12}[A, [A, B]] + \dots$$

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先求下式:

$$\begin{aligned} e^{-xB}e^{-xA} \frac{d}{dx} e^{xA}e^{xB} &= e^{-xB} B e^{xB} + e^{-xB} e^{-xA} A e^{xA} e^{xB} \\ &= B + e^{-xB} A e^{xB} \\ &= B + A + x[A, B] + \frac{x^2}{2}[B, [B, A]] + \dots \end{aligned}$$

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对  $G'(x) = G_1 + 2xG_2 + 3x^2G_3 + \dots$  展开, 再求下式:

$$\begin{aligned} e^{-G(x)} \frac{d}{dx} e^{G(x)} &= G' + \frac{1}{2}[G', G] + \frac{1}{3!}[[G', G], ] + \dots \\ &= G_1 + 2xG_2 + x^2 \left( 3G_3 - \frac{1}{3}[G_1, G_2] \right) + \dots \end{aligned}$$

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把两个式子对比一下:

$$\begin{aligned} e^{-xB}e^{-xA} \frac{d}{dx} e^{xA}e^{xB} &= B + A + x[A, B] + \frac{x^2}{2}[B, [B, A]] + \dots \\ e^{-G(x)} \frac{d}{dx} e^{G(x)} &= G_1 + 2xG_2 + x^2 \left( 3G_3 - \frac{1}{3[G_1, G_2]} \right) + \dots \end{aligned}$$

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得到

$$G_1 = A + B, \quad G_2 = \frac{1}{2}[A, B], \quad G_3 = \frac{1}{12}([A, [A, B]] + [B, [B, A]])$$

从而, 有

$$G(x) = x(A + B) + \frac{x^2}{2}[A, B] + \frac{x^3}{12}([A, [A, B]] + [B, [B, A]])$$

## 相干态的性质

$$\hat{a} |\alpha\rangle = \hat{a} e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} |0\rangle$$

- 定义  $\hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}$
- 满足:  $\hat{D}^\dagger(\alpha) = \hat{D}(-\alpha) = [\hat{D}(\alpha)]^{-1}$

$$\hat{D}^\dagger(\alpha) \hat{a} \hat{D}(\alpha) = \hat{a} + [-\alpha \hat{a}^\dagger + \alpha^* \hat{a}, \hat{a}] = \hat{a} + \alpha$$

$$\hat{D}^\dagger(\alpha) \hat{a}^\dagger \hat{D}(\alpha) = \hat{a}^\dagger + \alpha^*$$

得到  $\hat{a} \hat{D}(\alpha) = \hat{D}(\alpha) (\hat{a} + \alpha)$

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得到  $\hat{a} \hat{D}(\alpha) = \hat{D}(\alpha) (\hat{a} + \alpha)$

$$\begin{aligned} \hat{a} |\alpha\rangle &= \hat{a} \hat{D}(\alpha) |0\rangle \\ &= \hat{D}(\alpha) (\hat{a} + \alpha) |0\rangle \\ &= \alpha \hat{D}(\alpha) |0\rangle = \alpha |\alpha\rangle \end{aligned}$$

$|\alpha\rangle$  是算符  $\hat{a}$  的本征态

## 相干态的性质

- $\hat{a}$  不是厄米算符,  $\alpha$  不是实数

$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle, \quad \langle\alpha| \hat{a}^\dagger = \langle\alpha| \alpha^*$$

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$$\hat{a} |\alpha\rangle = \alpha |\alpha\rangle, \quad \langle\alpha| \hat{a}^\dagger = \langle\alpha| \alpha^*$$

- 粒子数表象

$$\begin{aligned} |\alpha\rangle &= \sum_n |n\rangle \langle n|\alpha\rangle = \sum_n |n\rangle \langle 0|\frac{\hat{a}^n}{\sqrt{n!}}|\alpha\rangle \\ &= \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \langle 0|\alpha\rangle \end{aligned}$$

归一化条件

$$\begin{aligned} 1 = \langle\alpha|\alpha\rangle &= \sum_m \frac{\alpha^{*m}}{\sqrt{m!}} \langle\alpha|0\rangle \langle m|n\rangle \frac{\alpha^n}{\sqrt{n!}} \langle 0|\alpha\rangle \\ &= \sum_n \frac{|\alpha|^{2n}}{n!} |\langle 0|\alpha\rangle|^2 = e^{|\alpha|^2} |\langle 0|\alpha\rangle|^2 \\ \rightarrow \langle 0|\alpha\rangle &= e^{-|\alpha|^2/2} \end{aligned}$$

得到

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

## 相干态性质

也可以表示为

$$\begin{aligned} |\alpha\rangle &= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \\ &= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0\rangle \\ &= e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} |0\rangle \end{aligned}$$

BCH 公式

$$\hat{D}(\alpha) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}} = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} = e^{|\alpha|^2/2} e^{-\alpha^* \hat{a}} e^{\alpha \hat{a}^\dagger}$$

# 相干态性质

平移算符

$$\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha, \quad \hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha) = \hat{a}^\dagger + \alpha^*$$

有以下性质

$$\begin{aligned}\hat{D}^\dagger(\alpha)F(\hat{a}, \hat{a}^\dagger)\hat{D}(\alpha) &= F(\hat{a} + \alpha, \hat{a}^\dagger + \alpha^*) \\ \hat{D}(\alpha)\hat{D}(\beta) &= \hat{D}(\alpha + \beta)e^{(\alpha\beta^* - \alpha^*\beta)/2}\end{aligned}$$

## 相干态性质: 非正交性

$$\begin{aligned}\langle\alpha|\beta\rangle &= \langle 0|\hat{D}(-\alpha)\hat{D}(\beta)|0\rangle \\ &= e^{(-\alpha\beta^* + \alpha^*\beta)/2} \langle 0|\hat{D}(\beta - \alpha)|0\rangle \\ &= \exp\left(\frac{-\alpha\beta + \alpha^*\beta}{2}\right) \exp\left(-\frac{1}{2}|\beta - \alpha|^2\right) \\ &= \exp\left(\frac{-\alpha\beta + \alpha^*\beta}{2}\right) \exp\left(-\frac{\alpha^*\alpha - \alpha^*\beta - \alpha\beta^* + \beta^*\beta}{2}\right) \\ &= \exp\left(-\frac{\alpha^*\alpha - 2\alpha^*\beta + \beta^*\beta}{2}\right)\end{aligned}$$

- 得到  $|\langle\alpha|\beta\rangle| = \exp(-|\alpha - \beta|^2)$
- 所有的相干态不正交
- 两个相干态的交叠与  $|\alpha - \beta|$  的大小有关

这个值越大, 两个态的交叠越小

## 相干态性质: 超完备性

$$J = \int d^2\alpha |\alpha\rangle\langle\alpha|$$

- $d^2\alpha = dx dy$ , 其中定义  $\alpha = x + iy$

## 相干态性质: 超完备性

$$\begin{aligned} J &= \int d^2\alpha |\alpha\rangle\langle\alpha| \\ &= \sum_{n,n'} \frac{|n\rangle\langle n'|}{\sqrt{n!n'!}} \int d^2\alpha (\alpha^*)^n \alpha^{n'} \exp(-|\alpha|^2) \end{aligned}$$

## 相干态性质: 超完备性

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- 积分变换:  $\alpha = r e^{i\theta}$

## 相干态性质: 超完备性

$$\begin{aligned} J &= \int d^2\alpha |\alpha\rangle\langle\alpha| \\ &= \sum_{n,n'} \frac{|n\rangle\langle n'|}{\sqrt{n!n'!}} \int d^2\alpha (\alpha^*)^n \alpha^{n'} \exp(-|\alpha|^2) \\ &= \sum_{n,n'} \frac{|n\rangle\langle n'|}{\sqrt{n!n'!}} \int dr d\theta r^{n+n'+1} e^{-r^2} e^{i(n'-n)\theta} \\ &= 2\pi \sum_n \frac{|n\rangle\langle n|}{n!} \int dr r^{2n+1} e^{-r^2} \\ &= 2\pi \sum_n \frac{|n\rangle\langle n|}{n!} \int dx x^n e^{-x} \\ &= \pi \sum_n |n\rangle\langle n| \end{aligned}$$

- 积分变换:  $\alpha = r e^{i\theta}$
- 积分变换:  $x = r^2$
- Gamma 函数  $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$

## 相干态性质：超完备性

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \mathbf{1}$$

$$\int \frac{d\alpha d\alpha^*}{2\pi i} |\alpha\rangle\langle\alpha| = \mathbf{1}$$

## 相干态性质：超完备性

- 所有的态矢量都非正交
- 数学的冗余

思考与正交基矢的区别

- 任意的态矢量在相干态表象的展开**不唯一**

## 相干态性质

对于任何态矢量  $|f\rangle$ , 可以得到

$$|f\rangle = \int d^2\alpha \frac{1}{\pi} |\alpha\rangle \langle\alpha|f\rangle = \int d^2\alpha f(\alpha, \alpha^*) |\alpha\rangle$$

## 相干态性质

例子：真空态  $|0\rangle$ ：一种情况  $f(\alpha) = \delta(\alpha)$ ，或者可以展开

$$|0\rangle = \int \frac{d^2\alpha}{\pi} \langle\alpha|0\rangle |\alpha\rangle = \int \frac{d^2\alpha}{\pi} e^{-|\alpha|^2/2} |\alpha\rangle$$

## 相干态性质

例子：真空态  $|0\rangle$ ：一种情况  $f(\alpha) = \delta(\alpha)$ ，或者可以展开

$$\begin{aligned} |0\rangle &= \int \frac{d^2\alpha}{\pi} \langle \alpha|0\rangle |\alpha\rangle = \int \frac{d^2\alpha}{\pi} e^{-|\alpha|^2/2} |\alpha\rangle \\ &= \int \frac{d^2\alpha}{\pi} e^{-|\alpha|^2/2} (1 + c_m \alpha^m) |\alpha\rangle \end{aligned}$$

Note that

$$\int d^2\alpha e^{-|\alpha|^2/2} \alpha^{m+n} = \int dr d\theta r e^{-r^2/2} r^{m+n} e^{i(m+n)\theta} = 0$$

所以

$$f(\alpha, \alpha^*) = \begin{cases} \delta(\alpha) \\ e^{-|\alpha|^2/2} \\ e^{-|\alpha|^2/2} (1 + c_m \alpha^m) \\ \dots \end{cases}$$

## 相干态性质

$$|f\rangle = \frac{1}{\pi} \int d^2\alpha |\alpha\rangle f(\alpha^*) \exp(-|\alpha|^2/2)$$

要求展开系数  $f(\alpha^*) = e^{|\alpha|^2/2} \langle \alpha | f \rangle$  是复平面内除  $\infty$  外处处解析的函数，并且满足

$$|f(\alpha^*)| \leq e^{|\alpha|^2/2} \| |f\rangle \|$$

可以证明：这样定义的函数  $f(\alpha^*)$  是唯一的

## 相干态性质

左乘  $\langle \alpha' |$ , 得到

$$\begin{aligned}\langle \alpha' | f \rangle &= \frac{1}{\pi} \int d^2\alpha \langle \alpha' | \alpha \rangle e^{-|\alpha|^2/2} f(\alpha^*) \\ &= \frac{1}{\pi} \int d^2\alpha e^{-|\alpha'|^2/2} e^{\alpha'^* \alpha - |\alpha|^2} f(\alpha^*) \\ &= \frac{1}{\pi} \int d^2\alpha e^{-|\alpha'|^2/2} e^{\alpha'^* \alpha - |\alpha|^2} \sum_n c_n (\alpha^*)^n \\ &= \frac{1}{\pi} \int r dr d\theta e^{-|\alpha'|^2/2} \sum_m \frac{(\alpha'^* r e^{i\theta})^m}{m!} e^{-r^2} \sum_n c_n r^n e^{-in\theta} \\ &= \frac{1}{\pi} \sum_{m,n} \int e^{-|\alpha'|^2/2} \frac{(\alpha'^*)^m c_n}{m!} r^{m+n+1} e^{-r^2} e^{i(m-n)\theta} dr d\theta \\ &= 2 \sum_n \int e^{-|\alpha'|^2/2} \frac{(\alpha'^*)^n c_n}{n!} r^{2n+1} e^{-r^2} dr \\ &= \sum_n e^{-|\alpha'|^2/2} c_n (\alpha'^*)^n \\ &= e^{-|\alpha'|^2/2} f(\alpha'^*)\end{aligned}$$

## 算符展开

$$\begin{aligned}\hat{T} &= I\hat{T}I = \frac{1}{\pi^2} \int d^2\alpha d^2\beta |\alpha\rangle \langle\alpha|\hat{T}|\beta\rangle \langle\beta| \\ &= \frac{1}{\pi^2} d^2\alpha d^2\beta |\alpha\rangle \langle\beta| T(\alpha^*, \beta) \exp\left(-\frac{|\alpha|^2 + |\beta|^2}{2}\right)\end{aligned}$$

- $T(\alpha^*, \beta)$  是复平面内  $\alpha^*$  和  $\beta$  的解析函数

# 相干态性质

## 平移算符的迹

$$\begin{aligned}\mathrm{Tr}[\hat{D}(\lambda)] &= \frac{1}{\pi} \int d^2\alpha \langle \alpha | \hat{D}(\lambda) | \alpha \rangle \\ &= \frac{1}{\pi} \int d^2\alpha \langle \hat{D}^\dagger(\alpha) \hat{D}(\lambda) \hat{D}(\alpha) \rangle \\ &= \frac{1}{\pi} \int d^2\alpha \langle 0 | e^{\lambda(\hat{a}^\dagger + \alpha^*) - \lambda^*(\hat{a} + \alpha)} | 0 \rangle \\ &= \frac{1}{\pi} \int d^2\alpha e^{\lambda\alpha^* - \lambda^*\alpha} e^{-|\lambda|^2/2} \\ &= \pi e^{-|\lambda|^2/2} \delta(\lambda_r) \delta(\lambda_i)\end{aligned}$$

其中  $\lambda = \lambda_r + i\lambda_i$

## 相干态性质

- 有时也会忽略前面的系数

$$\begin{aligned}\mathrm{Tr}[\hat{D}(\lambda)] &= \pi\delta^{(2)}(\lambda), \\ \mathrm{Tr}[\hat{D}(\alpha)\hat{D}^\dagger(\beta)] &= \pi\delta^{(2)}(\alpha - \beta)\end{aligned}$$

- 可以把  $\hat{D}(\alpha)$  作为一组基矢，展开任意算符  $\hat{O}$ :

$$\begin{aligned}\hat{O} &= \frac{1}{\pi} \int d^2\alpha O(\alpha)\hat{D}^\dagger(\alpha) \\ O(\alpha) &= \mathrm{Tr}[\hat{O}\hat{D}(\alpha)]\end{aligned}$$

## 泊松分布

- 假设光场中，光子出现的概率是随机的，在一段时间  $T$  内，出现的平均次数是  $\bar{n}$
- 把时间间隔  $T$  分成  $N$  份
- 小间隔足够短，最多只能出现一个光子
- 在每一个小的时间间隔出现光子的概率为  $p = \bar{n}/N$
- 统计光子在时间间隔  $T$  出现  $k$  次的概率是多少？

$$\begin{aligned} P(k) &= C_N^k p^k (1-p)^{N-k} \\ &= \frac{N(N-1)\dots(N-k+1)}{k!} \left(\frac{\bar{n}}{N}\right)^k \left(1 - \frac{\bar{n}}{N}\right)^{N-k} \\ &\approx \frac{\bar{n}^k}{k!} \left(1 - \frac{\bar{n}}{N}\right)^{N-k} \\ &\rightarrow \frac{\bar{n}^k}{k!} e^{-\bar{n}} \end{aligned}$$

泊松分布

## 相干态性质：光子数分布

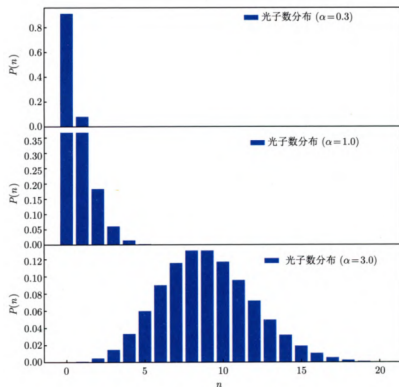
相干态  $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$  处在  $|k\rangle$  态的概率：

$$P_k(\alpha) = e^{-|\alpha|^2} \frac{|\alpha|^{2k}}{k!}$$

- 平均光子数为  $\bar{n} = |\alpha|^2$
- 光子数涨落  $\Delta n^2 = \bar{n} \rightarrow \frac{\sqrt{\Delta n^2}}{\bar{n}} = \frac{1}{\sqrt{\bar{n}}}$

当光子数很大，涨落可以忽略  $\rightarrow$  经典

## 相干态性质：光子数分布



不同相干态的光子数分布

## 相干态性质：不确定性关系

- 位置算符和动量算符 ( $x_{\text{zpf}} = \sqrt{\frac{\hbar}{2m\omega}}$  是基态位置的涨落)

$$\hat{x} = x_{\text{zpf}}(\hat{a} + \hat{a}^\dagger), \quad \hat{p} = \frac{\hbar}{2ix_{\text{zpf}}}(\hat{a} - \hat{a}^\dagger)$$

- 对于相干态  $|\alpha\rangle$ :

$$\langle \hat{x} \rangle = x_{\text{zpf}}(\alpha + \alpha^*), \quad \langle \hat{p} \rangle = \frac{\hbar}{2ix_{\text{zpf}}}(\alpha - \alpha^*)$$

$$\langle \hat{x}^2 \rangle = x_{\text{zpf}}^2(\alpha^{*2} + \alpha^2 + 2\alpha^*\alpha + 1), \quad \langle \hat{p}^2 \rangle = -\frac{\hbar^2}{4x_{\text{zpf}}^2}(\alpha^{*2} + \alpha^2 - 2\alpha^*\alpha - 1)$$

- 得到位置和动量的不确定性关系:  $\Delta x \Delta p = \hbar/2$

$$(\Delta x)^2 = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2 = x_{\text{zpf}}^2$$

$$(\Delta p)^2 = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2 = \frac{\hbar^2}{4x_{\text{zpf}}^2}$$

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- 相干态满足最小不确定条件:

相干态是量子力学所容许的最逼近经典极限的量子态

## 相干态性质：不确定性关系

定义算符  $\hat{a}$  的实部和虚部 (field quadratures, 场正交分量或者场的象限分量):

$$\hat{X}_1 = \frac{1}{2} (\hat{a} + \hat{a}^\dagger), \quad \hat{X}_2 = \frac{1}{2i} (\hat{a} - \hat{a}^\dagger)$$

- 对易关系  $[\hat{X}_1, \hat{X}_2] = \frac{i}{2}$ , 有不确定关系

$$(\Delta X_1)^2 (\Delta X_2)^2 \geq \frac{1}{16}$$

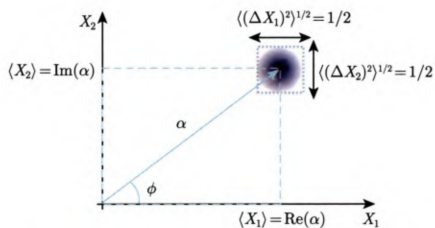
- 对于相干态, 可以得到

$$\begin{aligned} \langle \hat{X}_1 \rangle &= \text{Re}(\alpha), & \langle \hat{X}_2 \rangle &= \text{Im}(\alpha) \\ \langle \hat{X}_1^2 \rangle &= \frac{1}{4} + \langle \hat{X}_1 \rangle^2, & \langle \hat{X}_2^2 \rangle &= \frac{1}{4} + \langle \hat{X}_2 \rangle^2 \end{aligned}$$

从而

$$(\Delta X_1)^2 = (\Delta X_2)^2 = \frac{1}{4}$$

## 相干态性质：不确定性关系



- 真空态也是一个相干态：
- 平移算符  $\hat{D}(\alpha)$  不改变真空的涨落

相干态的量子涨落实际是真空的涨落

## 相干态性质：演化

谐振子系统，初态为  $|\alpha_0\rangle$ ，动力学演化

$$\begin{aligned} |\alpha(t)\rangle &= \exp\left(\frac{\hat{H}t}{i\hbar}\right) |\alpha_0\rangle = \exp(-i\omega\hat{a}^\dagger\hat{a}t) \exp(-|\alpha_0|^2/2) \sum_n \frac{\alpha_0^n}{\sqrt{n!}} |n\rangle \\ &= \sum_n \exp(-in\omega t - |\alpha_0|^2/2) \frac{\alpha_0^n}{\sqrt{n!}} |n\rangle \\ &= |\alpha_0 e^{-i\omega t}\rangle \end{aligned}$$

- 初态是相干态  $\rightarrow$  任意时刻都是相干态
- 动力学演化在相平面的轨迹是一个圆，半径是频率  $\omega$
- 期望

$$\langle \hat{X}_1 \rangle = \text{Re}\{\alpha(t)\} = (\alpha_0 e^{i\omega t} + \alpha_0^* e^{-i\omega t})/2$$

$$\langle \hat{X}_2 \rangle = \text{Im}\{\alpha(t)\} = (\alpha_0 e^{i\omega t} - \alpha_0^* e^{-i\omega t})/(2i)$$

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- 初态是相干态  $\rightarrow$  任意时刻都是相干态
- 动力学演化在相平面的轨迹是一个圆，半径是频率  $\omega$
- 期望 (设  $\alpha_0$  实数)

$$\langle \hat{X}_1 \rangle = \text{Re}\{\alpha(t)\} = \alpha_0 \cos(\omega t)$$

$$\langle \hat{X}_2 \rangle = \text{Im}\{\alpha(t)\} = -\alpha_0 \sin(\omega t)$$

oscillating just like a classical system

## 相干态性质：位置表象的波函数

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

- 把  $\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} + i\hat{p}) = \frac{1}{2x_{\text{zpf}}}\hat{x} + i\frac{x_{\text{zpf}}}{\hbar}\hat{p}$  代入上式
- 左乘  $\langle x|$ , 得到

$$\langle x|\frac{x}{2x_{\text{zpf}}} + \frac{\partial}{\partial(x/x_{\text{zpf}})}|\alpha\rangle = \alpha\langle x|\alpha\rangle$$

- 解微分方程：设  $a = x/x_{\text{zpf}}$ ,  $f(a) = \langle x|\alpha\rangle$

$$\begin{aligned}\frac{a}{2}f(a) + \frac{\partial}{\partial a}f(a) &= \alpha f(a) \\ \rightarrow f(a) &= \mathcal{N} \exp\left(-\frac{a^2}{4} + \alpha a\right)\end{aligned}$$

- 位置表象波函数

$$\langle x|\alpha\rangle = \mathcal{N} \exp\left(-\frac{x^2}{4x_{\text{zpf}}^2} + \frac{\alpha x}{x_{\text{zpf}}}\right)$$

## 相干态性质：位置表象的波函数

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

- 把  $\hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}(m\omega\hat{x} + i\hat{p}) = \frac{1}{2x_{\text{zpf}}}\hat{x} + i\frac{x_{\text{zpf}}}{\hbar}\hat{p}$  代入上式
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$$\begin{aligned}\frac{a}{2}f(a) + \frac{\partial}{\partial a}f(a) &= \alpha f(a) \\ \rightarrow f(a) &= \mathcal{N} \exp\left(-\frac{a^2}{4} + \alpha a\right)\end{aligned}$$

- 位置表象波函数

$$\begin{aligned}\langle x|\alpha\rangle &= (2\pi x_{\text{zpf}}^2)^{-1/4} \exp\left(-\frac{x^2}{4x_{\text{zpf}}^2} + \frac{\alpha x}{x_{\text{zpf}}} - \frac{(\alpha + \alpha^*)^2}{4} + i\varphi\right) \\ &= (2\pi x_{\text{zpf}}^2)^{-1/4} \exp\left(-\frac{(x - \langle x \rangle)^2}{4x_{\text{zpf}}^2} + i\frac{\langle p \rangle}{\hbar}x + i\varphi\right)\end{aligned}$$

什么是相干态？怎么理解相干态？

## 相干态性质

相干态是连接量子世界和经典世界的桥梁，它保持了量子力学基本性质的同时，最大限度地表现出经典物理的确定性和稳定性

- 最接近经典极限的量子态
- 满足最小不确定性关系

当粒子数很大时，有  $\frac{\Delta n}{\langle n \rangle} \rightarrow 0 \rightarrow$  粒子数涨落接近经典

- 光子数分布满足泊松分布
- 相空间轨迹：复振幅  $\alpha(t)$  在复平面的时间轨迹是一个圆  
经典轨迹
- 稳定性：在相干态湮灭一个光子，状态不改变
- 超完备性和非正交性

在相干态产生一个光子会怎么样

$$\hat{a}^\dagger |\alpha\rangle = ?$$

## 相干态性质

$$\begin{aligned}\hat{a}^\dagger |\alpha\rangle &= \hat{a}^\dagger \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \\ &= \exp\left(-\frac{1}{2}\alpha\alpha^*\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n+1} |n+1\rangle \\ &= \exp\left(-\frac{1}{2}\alpha\alpha^*\right) \frac{\partial}{\partial\alpha} \sum_{n=0}^{\infty} \frac{\alpha^{n+1}}{\sqrt{n+1}} |n+1\rangle \\ &= \exp\left(-\frac{1}{2}\alpha\alpha^*\right) \frac{\partial}{\partial\alpha} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n}} |n\rangle \\ &= \left(\frac{\alpha^*}{2} + \frac{\partial}{\partial\alpha}\right) \exp\left(-\frac{1}{2}\alpha\alpha^*\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n}} |n\rangle \\ &= \left(\frac{\alpha^*}{2} + \frac{\partial}{\partial\alpha}\right) |\alpha\rangle\end{aligned}$$

- Photon added coherent state
- 证明

$$\hat{a} |\alpha\rangle\langle\alpha| = \left(\alpha^* + \frac{\partial}{\partial\alpha}\right) |\alpha\rangle\langle\alpha|$$

为什么  $\hat{a}^\dagger$  没有本征态?

## 相干态性质：电磁场

$$\mathbf{A}(\mathbf{r}, t) = \sqrt{\frac{\hbar}{2\omega\epsilon_0}} (\hat{a} + \hat{a}^\dagger) \mathbf{u}(\mathbf{r})$$

- Fock 态中：  $\langle n | \mathbf{A} | n \rangle = 0$
- 相干态

$$\langle \alpha | \mathbf{A} | \alpha \rangle = \sqrt{\frac{\hbar}{2\omega\epsilon_0}} (\alpha + \alpha^*) \mathbf{u}(\mathbf{r})$$

经常用来对电磁场做平均场近似

## 相干态性质: Husimi-Q 分布函数

利用相干态的完备性, 定义任意量子态  $\rho$  在相空间的 Husimi-Q 分布函数

$$Q(\alpha, \alpha^*) = \frac{1}{\pi} \langle \alpha | \rho | \alpha \rangle$$

衡量量子态  $\rho$  与一个中心在  $\alpha$  的经典波包有多像

- $Q(\alpha, \alpha^*)$  是非负的
- 归一性

$$\int d^2\alpha Q(\alpha, \alpha^*) = 1$$

→ 相空间的概率分布

- 利用  $\hat{a} = (\hat{x} + i\hat{p})/\sqrt{2}$ , 通常也把  $Q(\alpha, \alpha^*)$  表示在  $x-p$  平面上

## 相干态性质: Husimi-Q 分布函数

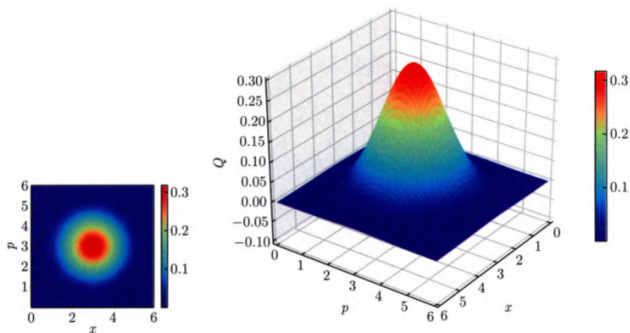
例子 1: 相干态  $\rho = |\beta\rangle\langle\beta|$  的 Q 函数

$$\begin{aligned}Q(\alpha, \alpha^*) &= \frac{1}{\pi} \langle\alpha|\beta\rangle \langle\beta|\alpha\rangle \\&= \frac{1}{\pi} |\langle\alpha|\beta\rangle|^2 \\&= \frac{1}{\pi} e^{-|\alpha-\beta|^2}\end{aligned}$$

利用  $\alpha = \frac{x+ip}{\sqrt{2}}$  转换到  $x$ - $p$  平面

$$\begin{aligned}Q(x, p) &= \frac{1}{\pi} e^{-\left|\frac{x+ip}{\sqrt{2}} - \beta\right|^2} \\&= \frac{1}{\pi} e^{-(x/\sqrt{2} - \beta_r)^2 - (p/\sqrt{2} - \beta_i)^2}\end{aligned}$$

## 相干态性质: Husimi-Q 分布函数



$\left| \beta = \frac{3+3i}{\sqrt{2}} \right\rangle$  的 Q 函数

## 相干态性质: Husimi-Q 分布函数

考虑一个“薛定谔猫态”

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\alpha\rangle + |-\alpha\rangle)$$

它的 Q 函数是什么样子

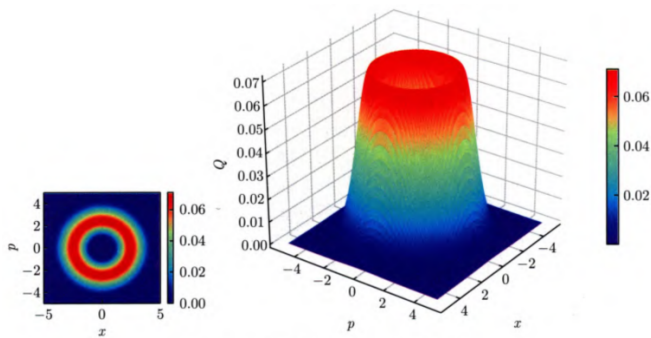
- 如何判断一个态的“量子化程度”

## 相干态性质: Husimi-Q 分布函数

例子 2: Fock 态  $\rho = |n\rangle\langle n|$

$$\begin{aligned} Q(\alpha, \alpha^*) &= \frac{1}{\pi} |\langle \alpha | n \rangle|^2 \\ &= e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} \\ &\rightarrow e^{-x^2/2 - p^2/2} \frac{1}{n!} \left( \frac{x^2}{2} + \frac{p^2}{2} \right)^n \end{aligned}$$

## 相干态性质: Husimi-Q 分布函数



$|n=3\rangle$  的 Q 函数

### 反正规排序算符的平均值

$$\begin{aligned}\langle \hat{a}^k \hat{a}^{\dagger l} \rangle &= \text{Tr}(\rho \hat{a}^k \hat{a}^{\dagger l}) \\ &= \int \frac{d^2\alpha}{\pi} \langle \alpha | \hat{a}^{\dagger l} \rho \hat{a}^k | \alpha \rangle \\ &= \int d^2\alpha \alpha^k (\alpha^*)^l Q(\alpha, \alpha^*)\end{aligned}$$

- 对于非反正规排序的算符  $\rightarrow$  利用对易关系, 化成反正规排序

## 相干态性质

$$|\alpha(t)\rangle = |\alpha_0 e^{-i\omega t}\rangle$$

- 相干来源于光学的经典相干性，在量子力学中表现为能级之间的相位关联
- 相位具有稳定性和确定性
- 演化：相位锁定，步调一致

动力学上形状不变，步调一致地集体迁移

## 相干态性质

$$|\alpha(t)\rangle = |\alpha_0 e^{-i\omega t}\rangle$$

- 激光 (laser): 激光的原理是受激辐射, 产生光子具有相同的频率、相位、和方向

激光的光子数很大, 也就是  $\alpha$  很大

- 超流 (superfluid): 使用一个相干态来表示超液态组成部分可以直接推导出大多数超液的特殊特性, 这个相干态在整个超液体积里拥有相同的波幅和相
- 超导 (superconductor): 电子是费米子, 但是它们成对形成库柏对后像玻色子一样反应, 能够在低温下一起形成相干态

## 相干态与路径积分

$$\begin{aligned} Z &= \text{Tr}(e^{-\beta H}) \\ &= \text{Tr}(e^{\delta\tau H} \dots e^{-\delta\tau H}) \\ &= \int \frac{d^2\alpha_0}{\pi} \langle \alpha_0 | e^{-\delta\tau H} \int \frac{d^2\alpha_{N-1}}{\pi} |\alpha_{N-1}\rangle \langle \alpha_{N-1}| \dots \int \frac{d^2\alpha_1}{\pi} e^{-\delta\tau H} |\alpha_0\rangle \\ &= \int \prod_{i=0}^{N-1} \frac{d^2\alpha_i}{\pi} \langle \alpha_{i+1} | e^{-\delta\tau H} | \alpha_i \rangle \end{aligned}$$

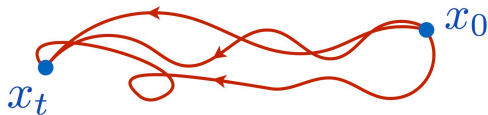
- 这里  $\delta\tau = \beta/N$ ,  $N \rightarrow +\infty$ , 约定  $\alpha_0 = \alpha_N$
- 玻色系统: 算符可以直接用复数  $\alpha$  或者  $\alpha^*$  代替

$$\langle \alpha_{i+1} | H(\hat{a}, \hat{a}^\dagger) | \alpha_i \rangle = H(\alpha_{i+1}^*, \alpha_i)$$

- 自动包含正规序
- 声子、磁振子等集体激发的玻色系统
- 强关联物理

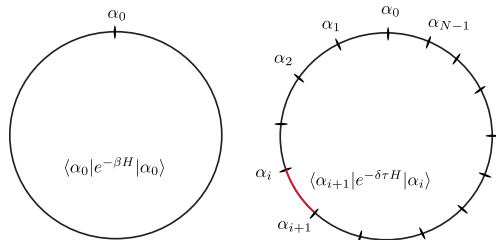
## 相干态与路径积分

$$\langle x_t | e^{-i\hat{H}t/\hbar} | x_0 \rangle =$$



路径积分思想：从  $x_0$  到  $x_t$ ，对所有路径积分

# 相干态与路径积分



用相干态表象计算路径积分

$$Z = \int \prod_{i=0}^{N-1} \frac{d^2\alpha_i}{\pi} \langle \alpha_{i+1} | e^{-\delta\tau H} | \alpha_i \rangle$$

算符到复数域的映射

## 第二章

相干态的性质

**压缩相干态**

光场的 Wigner 特征函数

自旋相干态/原子相干态

总结

## 压缩相干态

- 相干态满足最小不确定条件  $\Delta X_1 \Delta X_2 \geq 1/4$ , 并且  $\Delta X_1 = \Delta X_2 = 1/2$
- 有没有满足最小不确定条件, 但是

$$\Delta X_1 > \frac{1}{2}, \quad \Delta X_2 < \frac{1}{2}$$

的量子态

→ 提高探测精度

## 最小不确定态

假定两个算符  $\hat{A}$  和  $\hat{B}$  满足以下关系

$$[\hat{A}, \hat{B}] = i\hat{C}$$

定义  $|f\rangle = (\hat{A} - \langle A \rangle) |\psi\rangle$ ,  $|g\rangle = (\hat{B} - \langle B \rangle) |\psi\rangle$

- 不确定性  $\sigma_A^2 = \langle f|f\rangle$ ,  $\sigma_B^2 = \langle g|g\rangle$
- Cauchy-Schwarz inequality

$$\begin{aligned}\sigma_A^2 \sigma_B^2 &= \langle f|f\rangle \langle g|g\rangle \\ &\geq |\langle f|g\rangle|^2 \\ &= \left( \langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle \right) \left( \langle \hat{B}\hat{A} \rangle - \langle \hat{A} \rangle \langle \hat{B} \rangle \right) \\ &= \left( \frac{\langle \hat{A}\hat{B} \rangle + \langle \hat{B}\hat{A} \rangle - 2\langle \hat{A} \rangle \langle \hat{B} \rangle}{2} \right)^2 + \left( \frac{\langle \hat{A}\hat{B} \rangle - \langle \hat{B}\hat{A} \rangle}{2i} \right)^2 \\ &\implies \sigma_A \sigma_B \geq \frac{1}{2} |\langle \hat{C} \rangle|\end{aligned}$$

当  $|f\rangle = \lambda |g\rangle$  时, 上面的等号成立,  $\lambda$  为复数域的常数

## 最小不确定态

$$\text{令 } \hat{A} = \hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger), \hat{B} = \hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)$$

$$\begin{aligned} \left( \frac{\hat{a} + \hat{a}^\dagger}{2} - \langle \hat{X}_1 \rangle \right) |\psi\rangle &= \lambda \left( \frac{\hat{a} - \hat{a}^\dagger}{2i} - \langle \hat{X}_2 \rangle \right) |\psi\rangle \\ \left[ \left( \frac{1}{2} + \frac{i}{2}\lambda \right) \hat{a} + \left( \frac{1}{2} - \frac{i}{2}\lambda \right) \hat{a}^\dagger \right] |\psi\rangle &= \left( \langle \hat{X}_1 \rangle - \lambda \langle \hat{X}_2 \rangle \right) |\psi\rangle \end{aligned}$$

- 当  $\lambda = -i$ ,  $|\psi\rangle$  为  $\hat{a}$  的本征态, 也就是相干态  
标准相干态
- 当  $\lambda \neq -i$ : 仍然满足最小不确定性  $\rightarrow$  压缩相干态

## 压缩相干态

设  $\hat{b} = u\hat{a} + \nu\hat{a}^\dagger$ , 系数满足  $u^*u - \nu^*\nu = 1$ , 则  $\hat{b}$  也满足玻色算符的对易关系

$$[\hat{b}, \hat{b}^\dagger] = 1$$

## 压缩相干态

设  $\hat{b} = \cosh r \hat{a} + \sinh r \hat{a}^\dagger$ , 系数满足  $u^* u - \nu^* \nu = 1$ , 则  $\hat{b}$  也满足玻色算符的对易关系

$$[\hat{b}, \hat{b}^\dagger] = 1$$

- 玻戈留波夫变换 (Bogoliubov transformation)  
BCS 理论、霍金辐射、核物理配对效应
- 准粒子算符

$$\hat{N} = \hat{b}^\dagger \hat{b} = (u^* u + \nu^* \nu) \hat{a}^\dagger \hat{a} + u^* \nu \hat{a}^\dagger \hat{a}^\dagger + \nu^* u \hat{a} \hat{a} + \nu^* \nu$$

- 这样的量子态有什么性质？

$$\exp\left(-i \frac{g \hat{a} \hat{a} + g^* \hat{a}^\dagger \hat{a}^\dagger}{\hbar} t\right) |0\rangle = ?$$

## 压缩相干态

设  $\hat{b} = \cosh r \hat{a} + \sinh r \hat{a}^\dagger$ , 系数满足  $u^* u - \nu^* \nu = 1$ , 则  $\hat{b}$  也满足玻色算符的对易关系

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- 玻戈留波夫变换 (Bogoliubov transformation)  
BCS 理论、霍金辐射、核物理配对效应
- 准粒子算符

$$\hat{N} = \hat{b}^\dagger \hat{b} = (u^* u + \nu^* \nu) \hat{a}^\dagger \hat{a} + u^* \nu \hat{a}^\dagger \hat{a}^\dagger + \nu^* u \hat{a} \hat{a} + \nu^* \nu$$

- 这样的量子态有什么性质？

$$\exp\left(\frac{\xi}{2} \hat{a}^2 - \frac{\xi}{2} \hat{a}^{\dagger 2}\right) |0\rangle = ?$$

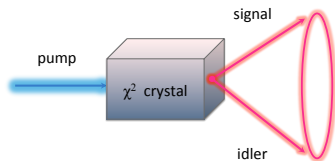
## 压缩相干态

- 介质中电场  $D = \epsilon_0 \mathbf{E} + \mathbf{P}$ , 电场的波动方程

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\epsilon_0 c^2} \frac{\partial^2 \mathbf{P}}{\partial t^2}$$

- 介质中有线性的极化, 也有非线性极化

$$P_i^{\text{NL}} = \epsilon_0 \sum_{j,k} \chi_{ijk}^{(2)} E_j E_k$$

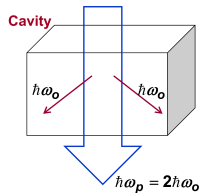


参量下转换 (Spontaneous parametric down conversion, SPDC)

$$\omega_p \rightarrow \omega_i + \omega_s$$

# 压缩相干态

哈密顿量



SPDC in cavity

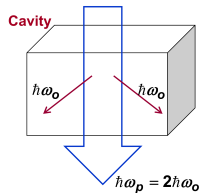
$$\begin{aligned}\hat{H} &= H_0 - \int_v \mathbf{P}^{\text{NL}} \cdot \mathbf{E} dV \\ &= \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar\omega_p \hat{a}_p^\dagger \hat{a}_p - \int d^3\mathbf{r} \epsilon_0 \chi^{(2)} \hat{\mathbf{E}}(\mathbf{r}) \hat{\mathbf{E}}(\mathbf{r}) \hat{\mathbf{E}}_p(\mathbf{r})\end{aligned}$$

$$\hat{\mathbf{E}}(\mathbf{r}) = i\sqrt{\frac{\hbar\omega_0}{2\epsilon_0\epsilon}} \left( \hat{a}\mathbf{u}(\mathbf{r}) - \hat{a}^\dagger\mathbf{u}^*(\mathbf{r}) \right)$$

$$\hat{\mathbf{E}}_p(\mathbf{r}) = i\sqrt{\frac{\hbar\omega_p}{2\epsilon_0\epsilon}} \left( \hat{a}_p\mathbf{u}_p(\mathbf{r}) - \hat{a}_p^\dagger\mathbf{u}_p^*(\mathbf{r}) \right)$$

# 压缩相干态

哈密顿量



SPDC in cavity

$$\hat{H} = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \hbar\omega_p \hat{a}_p^\dagger \hat{a}_p + \frac{i\hbar}{2} \left( \beta^* \hat{a}_p^\dagger \hat{a}^2 - \beta \hat{a}_p \hat{a}^{\dagger 2} \right)$$

- 假设驱动光处在相干态，做近似

$$\left\langle \alpha_p e^{-i\omega_p t} \left| \hat{a}_p \right| \alpha_p e^{-i\omega_p t} \right\rangle = \alpha_p e^{-i\omega_p t}$$

$$\left\langle \alpha_p e^{-i\omega_p t} \left| \hat{a}_p^\dagger \right| \alpha_p e^{-i\omega_p t} \right\rangle = \alpha_p^* e^{i\omega_p t}$$

## 压缩相干态

- 初态  $|\psi(t=0)\rangle = |0\rangle \otimes |\alpha_p\rangle$
- 使用相互作用表象, 设  $|\psi(t)\rangle = e^{-iH_0t/\hbar} |\phi(t)\rangle$ :

$$H_0 = \hbar\omega_0 \hat{a}^\dagger + \hbar\omega_p \hat{a}_p^\dagger \hat{a}_p, \quad H_I = \frac{i\hbar}{2} \left( \beta^* \hat{a}_p^\dagger \hat{a}^2 - \beta \hat{a}_p \hat{a}^{\dagger 2} \right)$$

- 波函数演化

$$|\phi(t)\rangle = e^{-iH_I t/\hbar} |0\rangle \otimes |\alpha_p\rangle$$

- 回到薛定谔表象

$$\begin{aligned} |\psi(t)\rangle &= e^{-iH_0t/\hbar} |\phi(t)\rangle = e^{-iH_0t/\hbar} e^{-iH_I t/\hbar} |0\rangle \otimes |\alpha_p\rangle \\ &= e^{-\frac{i}{\hbar} H_0 t} e^{-\frac{i}{\hbar} H_I t} e^{\frac{i}{\hbar} H_0 t} e^{-\frac{i}{\hbar} H_0 t} |0\rangle \otimes |\alpha_p\rangle \\ &= e^{-\frac{i}{\hbar} H_I t} e^{-\frac{i}{\hbar} H_0 t} |0\rangle \otimes |\alpha_p\rangle \\ &= e^{\frac{i}{2} (\beta^* \hat{a}_p^\dagger \hat{a}^2 - \beta \hat{a}_p \hat{a}^{\dagger 2})} |0\rangle \otimes \left| \alpha_p e^{-2i\omega_0 t} \right\rangle \\ &= e^{\frac{\xi^*(t)}{2} \hat{a}^2 - \frac{\xi(t)}{2} \hat{a}^{\dagger 2}} |0\rangle \otimes \left| \alpha_p e^{-2i\omega t} \right\rangle \end{aligned}$$

Assume  $\omega_p = 2\omega_0$

## 压缩相干态

双曲正弦、双曲余弦函数

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

泰勒展开

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

## 压缩相干态

定义压缩算符

$$\hat{S}(\xi) = \exp\left(\frac{\xi^*}{2}\hat{a}^2 - \frac{\xi}{2}\hat{a}^{\dagger 2}\right), \quad \xi = r \exp(i\theta)$$

$r$  为压缩参数,  $\theta$  为压缩角

- 么正性

$$\hat{S}^\dagger(\xi) = \hat{S}^{-1}(\xi) = \hat{S}(-\xi)$$

- 分解成压缩加旋转操作

$$\hat{S}(\xi) = e^{\frac{i}{2}\theta\hat{a}^\dagger\hat{a}} e^{\frac{r}{2}(\hat{a}^2 - \hat{a}^{\dagger 2})} e^{-\frac{i}{2}\theta\hat{a}^\dagger\hat{a}}$$

- 对湮灭算符和产生算符的作用

$$\begin{aligned}\hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi) &= \hat{a} \cosh r - \hat{a}^\dagger e^{i\theta} \sinh r \\ \hat{S}^\dagger(\xi)\hat{a}^\dagger\hat{S}(\xi) &= \hat{a}^\dagger \cosh r - \hat{a} e^{i\theta} \sinh r\end{aligned}$$

# 压缩相干态

## 压缩真空态

$$\begin{aligned} |0, \xi\rangle &= \hat{S}(\xi) |0\rangle = \exp\left(\frac{i}{2}\theta\hat{a}^\dagger\hat{a}\right) \exp\left[\frac{r}{2}(\hat{a}^2 - \hat{a}^{\dagger 2})\right] |0\rangle \\ &= \sqrt{\operatorname{sech} r} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{n!2^n} \left(-e^{i\theta} \tanh r\right)^n |2n\rangle \end{aligned}$$

- 只有粒子数为偶数的态有贡献

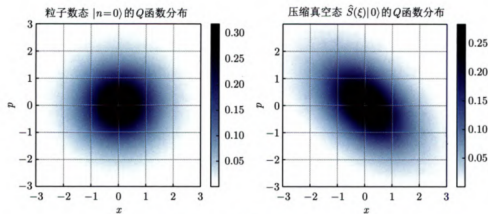
可以得到压缩真空态以下性质

$$\begin{aligned}\langle \hat{a} \rangle &= 0 \\ \langle \hat{a}^2 \rangle &= -\cosh r \sinh r e^{i\theta} \\ \langle \hat{a}^\dagger \hat{a} \rangle &= \sinh^2 r\end{aligned}$$

# 压缩相干态

## 压缩真空态内积

$$\langle 0, \xi | 0, \xi' \rangle = \left[ \frac{\operatorname{sech} r \operatorname{sech} r'}{1 - e^{i(\theta - \theta')} \tanh r \tanh r'} \right]^{1/2}$$



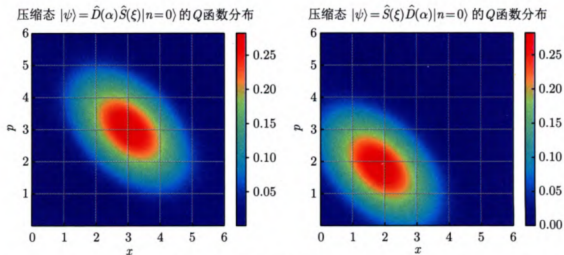
压缩真空态的 Q 函数，其中  $\xi = 0.5i$

# 压缩相干态

## 一般压缩态

$$|\alpha, \xi\rangle = \hat{D}(\alpha) |0, \xi\rangle = \hat{D}(\alpha) \hat{S}(\xi) |0\rangle$$

- 注意  $\hat{S}(\xi) \hat{D}(\alpha) \neq \hat{D}(\alpha) \hat{S}(\xi)$
- 有的教材也会定义  $|\alpha, \xi\rangle = \hat{S}(\xi) \hat{D}(\alpha) |0\rangle$



两种不同压缩态的 Q 函数,  $\alpha = \frac{3+3i}{\sqrt{2}}$ ,  $\xi = 0.5i$

## 压缩相干态

$$H = \omega \hat{a}^\dagger \hat{a} + \omega \hat{a} \hat{a}^\dagger + \Omega(\hat{a}^2 + \hat{a}^{\dagger 2})$$

- 基矢  $\Psi = \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix}$  ← Nambu spinor

$$H = \begin{pmatrix} \hat{a}^\dagger & \hat{a} \end{pmatrix} \begin{pmatrix} \omega & \Omega \\ \Omega & \omega \end{pmatrix} \begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} = \Psi^\dagger h \Psi$$

- 做线性变换:  $\hat{b} = u\hat{a} + \nu\hat{a}^\dagger$ , 为保证  $\hat{b}$  满足对易关系, 需要  $u^2 - \nu^2 = 1$

$$\hat{a} = u\hat{b} - \nu\hat{b}^\dagger, \quad \hat{a}^\dagger = u\hat{b}^\dagger - \nu\hat{b}$$

假定  $u, \nu$  为实数

- 哈密顿量变为

$$H = [\omega(u^2 + \nu^2) - 2\Omega u\nu] (\hat{b}\hat{b}^\dagger + \hat{b}^\dagger\hat{b}) - [(2\omega u\nu - \Omega(u^2 + \nu^2)) \hat{b}^{\dagger 2} + \text{H. c.}]$$

## 压缩相干态

当  $2\omega uv - \Omega(u^2 + v^2) = 0$  时，矩阵完全对角化，此时

$$u = \cosh r = \pm \sqrt{\frac{\omega}{2\varepsilon} + \frac{1}{2}}$$
$$v = \sinh r = \pm \sqrt{\frac{\omega}{2\varepsilon} - \frac{1}{2}}$$

其中  $\varepsilon = \sqrt{\omega^2 - \Omega^2}$

哈密顿量

$$\tilde{h} = \varepsilon(\hat{b}^\dagger \hat{b} + \hat{b} \hat{b}^\dagger)$$

# 压缩相干态

## 一般压缩态

$$\begin{aligned} |\alpha, \xi\rangle &= \hat{D}(\alpha)\hat{S}(r) |0\rangle \\ &= \hat{D}(\beta\hat{b}^\dagger - \beta^*\hat{b}) |0\rangle_r \end{aligned}$$

其中  $\beta = \mu\alpha + \nu\alpha^*$

- 真空压缩态是  $\hat{b}$  的基态

$$\hat{b} |0\rangle_r = 0, \quad \hat{b}^\dagger \hat{b} |0\rangle_g = 0$$

- 一般压缩态是  $\hat{b}$  的本征态

$$\hat{b} |\beta\rangle_g = \beta |\beta\rangle_g$$

- 如果  $\theta \neq 0$ , 可以做变换

$$u \rightarrow \cosh r, \quad \nu \rightarrow e^{i\theta} \sinh r$$

# 压缩相干态

## 一般压缩态

$$\begin{aligned} |\alpha, \xi\rangle &= \hat{D}(\alpha)\hat{S}(r)|0\rangle \\ &= \hat{D}(\beta\hat{b}^\dagger - \beta^*\hat{b})|0\rangle_r \end{aligned}$$

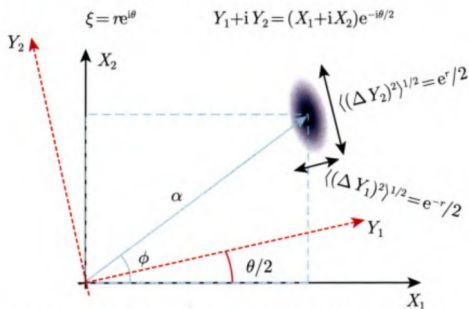
其中  $\beta = \mu\alpha + \nu\alpha^*$

$$\langle\hat{a}\rangle = {}_g\langle\beta|\hat{a}|\beta\rangle_g = u^*\beta - \nu\beta^* \equiv \alpha$$

$$\begin{aligned} \langle\hat{a}^\dagger\hat{a}\rangle &= {}_g\langle\beta|(u\hat{b}^\dagger - \nu^*\hat{b})(u^*\hat{b} - \nu\hat{b}^\dagger)|\beta\rangle_g \\ &= (u\beta^* - \nu^*\beta)(u^*\beta - \nu\beta^*) + \nu^*\nu = |\alpha|^2 + |\nu|^2 \end{aligned}$$

$$\langle\hat{a}^2\rangle = \alpha^2 - u\nu = \alpha^2 - \frac{1}{2}e^{i\theta} \sinh 2r$$

## 压缩相干态：涨落



$$\hat{Y}_1 = \cos \frac{\theta}{2} \hat{X}_1 + \sin \frac{\theta}{2} \hat{X}_2, \quad \hat{Y}_2 = -\sin \frac{\theta}{2} \hat{X}_1 + \cos \frac{\theta}{2} \hat{X}_2$$

$$\rightarrow \hat{Y}_1 + i \hat{Y}_2 = (\hat{X}_1 + i \hat{X}_2) e^{-i\theta/2} = \hat{a} e^{-i\theta/2}$$

- 压缩态在相空间中呈椭圆形
- 各向异性

## 压缩相干态：涨落

定义任意角度的正交相位振福算符

$$\hat{X}_\phi = \frac{1}{2} (\hat{a}e^{-i\phi} + \hat{a}^\dagger e^{i\phi}) = \hat{X}_1 \cos \phi + \hat{X}_2 \sin \phi$$

得到

$$\langle \alpha, \xi | \hat{X}_\phi | \alpha, \xi \rangle = \frac{1}{2} (\alpha e^{-i\phi} + \alpha^* e^{i\phi}), \quad \alpha = \langle \hat{X}_1 \rangle + i \langle \hat{X}_2 \rangle$$

$$\langle \alpha, \xi | \hat{X}_\phi^2 | \alpha, \xi \rangle = \frac{1}{4} [e^{2r} \sin^2(\phi - \theta/2) + e^{-2r} \cos^2(\phi - \theta/2)] + \langle \hat{X}_\phi \rangle^2$$

从而

$$\begin{aligned} (\Delta \hat{X}_\phi)^2 &= \frac{1}{4} [e^{2r} \sin^2(\phi - \theta/2) + e^{-2r} \cos^2(\phi - \theta/2)] \\ &= \frac{1}{4} \left| \cosh r - e^{i(2\phi - \theta)} \sinh r \right| \end{aligned}$$

- 当  $\phi = \theta/2$  时，涨落最小  $\Delta X_\phi = e^{-r}/2$
- 当  $\phi = \theta/2 + \pi/2$  时，涨落最大  $\Delta X_\phi = e^r/2$

## 压缩相干态

- 对于一对夹角相差  $\pi/2$  的正交相位算符

$$\Delta X_{\phi} \Delta X_{\phi+\pi/2} = \frac{1}{4} \left| 1 + (1 - e^{2i(2\phi-\theta)}) \sinh r \right| \geq \frac{1}{4}$$

- 等号成立条件为

$$\phi = \theta/2, \quad \theta = \frac{\pi}{2}$$

## 压缩相干态：涨落

电场

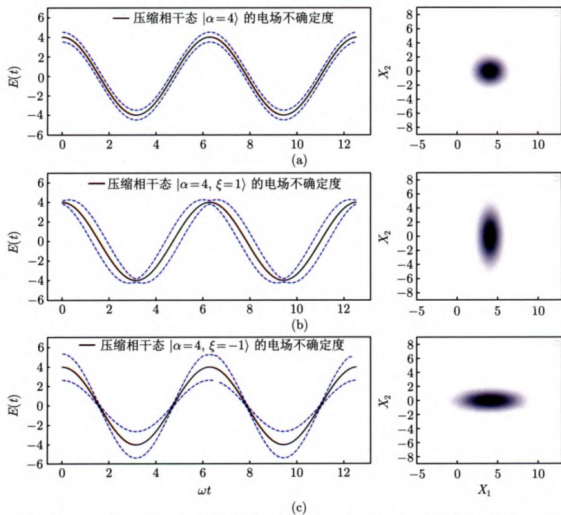
$$\begin{aligned}\hat{E} &= i\sqrt{\frac{\hbar\omega}{2\epsilon V}} \left( \hat{a}e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}} - \text{H. c.} \right) \\ &= 2\sqrt{\frac{\hbar\omega}{2\epsilon V}} \left[ \hat{X}_1 \sin(\omega t - \mathbf{k}\cdot\mathbf{r}) - \hat{X}_2 \cos(\omega t - \mathbf{k}\cdot\mathbf{r}) \right]\end{aligned}$$

电场的平均值和涨落

$$\begin{aligned}\langle \hat{E} \rangle &= 2\sqrt{\frac{\hbar\omega}{2\epsilon V}} \left[ \langle \hat{X}_1 \rangle \sin(\omega t - \mathbf{k}\cdot\mathbf{r}) - \langle \hat{X}_2 \rangle \cos(\omega t - \mathbf{k}\cdot\mathbf{r}) \right] \\ (\Delta E)^2 &= \frac{2\hbar\omega}{\epsilon V} \left[ (\Delta X_1)^2 \sin^2(\omega t - \mathbf{k}\cdot\mathbf{r}) + (\Delta X_2)^2 \cos^2(\omega t - \mathbf{k}\cdot\mathbf{r}) \right. \\ &\quad \left. - \sin(2\omega t - 2\mathbf{k}\cdot\mathbf{r}) V(\hat{X}_1, \hat{X}_2) \right]\end{aligned}$$

- $V(\hat{X}_1, \hat{X}_2) = \langle \hat{X}_1 \hat{X}_2 + \hat{X}_2 \hat{X}_1 \rangle / 2 - \langle \hat{X}_1 \rangle \langle \hat{X}_2 \rangle$
- 相干态和压缩相干态中  $V(\hat{X}_1, \hat{X}_2) = 0$
- 对于相干态  $\Delta X_1 = \Delta X_2$ , 所以涨落  $\Delta E$  是常数

# 压缩相干态：涨落



## 压缩相干态：粒子数分布

$$\langle n|\beta, \xi\rangle = \frac{(e^{i\theta/2} \tanh r)^{n/2}}{2^{n/2} \sqrt{n!} \cosh r} \exp\left[-\frac{1}{2}(\beta^{*2} e^{i\theta} \tanh r + |\beta|^2)\right] \\ \times H_n\left(\frac{\beta e^{-i\theta/2} \cosh r + \beta^* e^{i\theta/2} \sinh r}{\sqrt{2 \sinh r \cosh r}}\right)$$

- $H_n(z)$  为  $n$  阶厄米多项式

$$\langle \hat{N} \rangle = |\beta|^2 + \sinh^2 r$$

$$\langle \hat{N}^2 \rangle = (|\beta|^2 + \sinh^2 r)^2 + 2 \sinh^2 r \cosh^2 r + |\beta \cosh r - \beta e^{i\theta} \sinh r|^2$$

得到

$$\langle \Delta \hat{N}^2 \rangle = |\beta|^2 [\cosh 2r - \cos(\theta - 2\phi) \sinh 2r] + 2 \sinh^2 r \cosh^2 r$$

## 压缩相干态

### Mandel 参数

The **Mandel Q parameter** measures the departure of the occupation number distribution from Poissonian statistics

$$\mathcal{M} = \frac{\langle \Delta \hat{N}^2 \rangle - \langle \hat{N} \rangle}{\langle \hat{N} \rangle} = \langle \hat{N} \rangle (g^{(2)}(0) - 1)$$

- $\mathcal{M} = 0$ : 泊松分布
- $\mathcal{M} > 0$ : 超泊松分布
- $\mathcal{M} < 0$ : 亚泊松分布

## 压缩相干态：粒子数分布

- 压缩真空态： $\beta = 0$

$$\langle \Delta \hat{N}^2 \rangle = 2 \cosh^2 r \cosh^2 r \geq \langle \hat{N} \rangle = \sinh^2 r$$

### 超泊松分布

- 压缩角  $\theta = 2\phi$  时，压缩方向与相干振幅方向一致

$$\langle \Delta \hat{N}^2 \rangle = |\beta|^2 e^{-2r} + 2 \sinh^2 r \cosh^2 r$$

若相干振幅远大于压缩参数  $|\beta| \geq e^{2r}$ ,

$$\mathcal{M} \sim e^{-2r} - 1 < 0$$

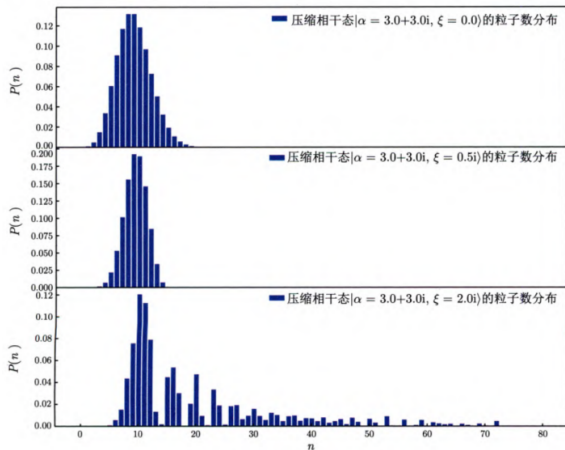
### 亚泊松分布

- 压缩角  $\theta = 2\phi + \pi$  时，压缩方向与相干振幅方向垂直

$$\mathcal{M} \sim e^{2r} - 1 > 0$$

### 超泊松分布

## 压缩相干态：粒子数分布



- $r$  增大，粒子数分布出现振荡
- $r$  增大到一定程度，不管压缩角在哪个方向，都会表现超泊松分布

## 压缩相干态：时间演化

哈密顿量  $H = \hbar\omega\hat{a}^\dagger\hat{a}$ ，初态为  $|\psi(t=0)\rangle = |\alpha, \xi\rangle = \hat{D}(\alpha)\hat{S}(\xi)|0\rangle$ :

$$\begin{aligned} |\psi(t)\rangle &= e^{-iHt/\hbar} |\psi(t=0)\rangle \\ &= e^{-iHt/\hbar} \hat{D}(\alpha)\hat{S}(\xi)|0\rangle \\ &= e^{-iHt/\hbar} \hat{D}(\alpha)e^{iHt/\hbar} e^{-iHt/\hbar} \hat{S}(\xi)e^{iHt/\hbar} e^{-iHt/\hbar} |0\rangle \\ &= \hat{D}(\alpha(t))\hat{S}(\xi(t))|0\rangle \\ &= |\alpha(t), \xi(t)\rangle \end{aligned}$$

where  $\alpha(t) = \alpha e^{i\omega t}$  and  $\xi(t) = \xi e^{-2i\omega t}$

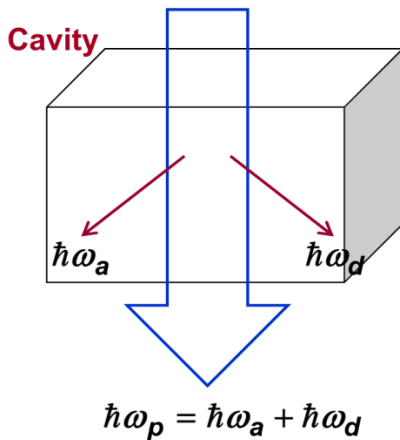
## 压缩相干态

- 同样具有稳定性，相位锁定
- 提高测量精度

引力波探测 [LIGO (美国), Virgo (意大利)]

- 连续变量量子信息处理
- 高灵敏度光谱学
- 产生纠缠源

## 双模压缩态



$$H = \hbar\omega_a \hat{a}^\dagger \hat{a} + \hbar\omega_d \hat{d}^\dagger \hat{d} + i\hbar\kappa \left( e^{i\omega_p t - i\theta_p} \hat{a} \hat{d} - e^{-i\omega_p t + i\theta_p} \hat{a}^\dagger \hat{d}^\dagger \right)$$

## 双模压缩态

考虑两个独立的模式

$$[\hat{a}_1, \hat{a}_1^\dagger] = [\hat{a}_2, \hat{a}_2^\dagger] = 1, \quad [\hat{a}_1, \hat{a}_2] = [\hat{a}_1, \hat{a}_2^\dagger] = 0$$

双模压缩算符

$$\hat{S}_{12}(\xi) = \exp\left(\xi^* \hat{a}_1 \hat{a}_2 - \xi \hat{a}_1^\dagger \hat{a}_2^\dagger\right)$$

## 双模压缩态

做变换

$$\hat{c} = \frac{\hat{a}_1 + \hat{a}_2}{\sqrt{2}}, \quad \hat{d} = \frac{\hat{a}_1 - \hat{a}_2}{\sqrt{2}}$$

双模压缩算符

$$\begin{aligned}\hat{S}_{12}(\xi) &= \exp\left(\xi^* \frac{\hat{c} + \hat{d}}{\sqrt{2}} \frac{\hat{c} - \hat{d}}{\sqrt{2}} - \xi \frac{\hat{c}^\dagger + \hat{d}^\dagger}{\sqrt{2}} \frac{\hat{c}^\dagger - \hat{d}^\dagger}{\sqrt{2}}\right) \\ &= \exp\left(\xi^* \hat{c}^2/2 - \xi \hat{c}^{\dagger 2}/2\right) \exp\left(-\xi^* \hat{d}^2/2 + \xi \hat{d}^{\dagger 2}/2\right) \\ &= \hat{S}_c(\xi) \hat{S}_d(-\xi)\end{aligned}$$

- 双模压缩态：两个独立模式压缩真空态的直积
- 一般双模压缩态

$$|\alpha, \beta, \xi\rangle = \hat{D}(\alpha) \hat{D}(\beta) \hat{S}_{12}(\xi) |0, 0\rangle$$

## 奇偶相干态

- 薛定谔猫态
- 叠加相干态  $|\psi\rangle = |\alpha\rangle + e^{i\phi} |-\alpha\rangle$
- 当  $\alpha \gg 1$  时，这样的态非常不稳定，寿命短，因此很难观测到，实际可能观测到的是混态

$$\rho \sim |\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|$$

- 当  $\alpha$  较小时，自由度相对较小，可以实现一些特定的量子态

## 奇偶相干态

$$\begin{aligned} |\alpha\rangle_o &= \frac{1}{\mathcal{N}_o} (|\alpha\rangle - |-\alpha\rangle) \\ &= \frac{1}{\mathcal{N}_o} \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{n=0}^{\infty} \left[ \frac{\alpha^n}{\sqrt{n!}} - \frac{(-\alpha)^n}{\sqrt{n!}} \right] |n\rangle \\ &= \frac{1}{\mathcal{N}_o} \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{n=0}^{\infty} \frac{2\alpha^{2n+1}}{\sqrt{(2n+1)!}} |2n+1\rangle \\ &= \frac{1}{\sqrt{\sinh|\alpha|^2}} \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{\sqrt{(2n+1)!}} |2n+1\rangle \end{aligned}$$

## 奇偶相干态

$$\begin{aligned} |\alpha\rangle_e &= \frac{1}{\mathcal{N}_e} (|\alpha\rangle + |-\alpha\rangle) \\ &= \frac{1}{\mathcal{N}_e} \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{n=0}^{\infty} \left[ \frac{\alpha^n}{\sqrt{n!}} + \frac{(-\alpha)^n}{\sqrt{n!}} \right] |n\rangle \\ &= \frac{1}{\mathcal{N}_e} \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{n=0}^{\infty} \frac{2\alpha^{2n}}{\sqrt{(2n)!}} |2n\rangle \\ &= \frac{1}{\sqrt{\cosh |\alpha|^2}} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{\sqrt{(2n)!}} |2n\rangle \end{aligned}$$

## 奇偶相干态

- 奇、偶相干态分别对应相干态奇数和偶数激发部分，它们必定正交

$${}_o\langle\alpha|\beta\rangle_e = 0, \quad {}_e\langle\alpha|\beta\rangle_o = 0$$

- 奇、偶相干态时湮灭算符  $\hat{a}^2$  的本征态

$$\hat{a}^2 |\alpha\rangle_o = \alpha^2 |\alpha\rangle_o, \quad \hat{a}^2 |\alpha\rangle_e = \alpha^2 |\alpha\rangle_e$$

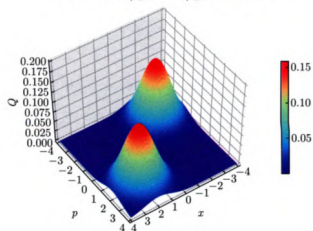
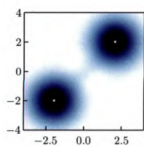
- 奇、偶相干态通过湮灭算符连接

$$\hat{a} |\alpha\rangle_o = \alpha \sqrt{\coth |\alpha|^2} |\alpha\rangle_e$$

$$\hat{a} |\alpha\rangle_e = \alpha \sqrt{\tanh |\alpha|^2} |\alpha\rangle_o$$

# 奇偶相干态

偶相干态  $|\psi\rangle \sim \left| \frac{2+2i}{\sqrt{2}} \right\rangle + \left| -\frac{2+2i}{\sqrt{2}} \right\rangle$  的 Q 函数分布



偶相干态  $|\alpha\rangle + |-\alpha\rangle$  的 Q 函数, 其中  $\alpha = \frac{2+2i}{\sqrt{2}}$

## 奇偶相干态

奇相干态和偶相干态只占了 Fock 空间一半基矢，它们单独不是完备的

$$I = \frac{1}{\pi} \int d^2\alpha e^{-|\alpha|^2} [\sinh |\alpha|^2 |\alpha\rangle_o \langle\alpha| + \cosh |\alpha|^2 |\alpha\rangle_e \langle\alpha|]$$

## 奇偶相干态：涨落特性

奇偶相干态是算符  $\hat{a}^2$  的本征态，一般会考察  $\hat{a}^2$  的实部和虚部

$$\hat{Y}_1 = \frac{\hat{a}^2 + (\hat{a}^\dagger)^2}{2}, \quad \hat{Y}_2 = \frac{\hat{a}^2 - (\hat{a}^\dagger)^2}{2i}$$

- 满足对易关系

$$[\hat{Y}_1, \hat{Y}_2] = i(2\hat{N} + 1)$$

- 其对应不确定关系为  $\Delta Y_1 \Delta Y_2 \geq \langle \hat{N} + \frac{1}{2} \rangle$ 。
- 可以得到

$$\begin{aligned}(\Delta Y_1)_o^2 &= (\Delta Y_2)_o^2 = N_o + \frac{1}{2} \\(\Delta Y_1)_e^2 &= (\Delta Y_2)_e^2 = N_e + \frac{1}{2}\end{aligned}$$

也是最小不确定态

## 第二章

相干态的性质

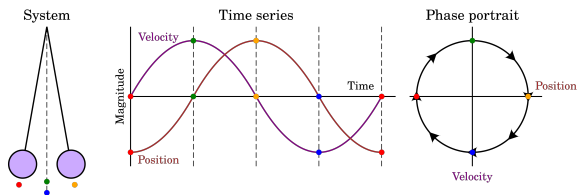
压缩相干态

光场的 Wigner 特征函数

自旋相干态/原子相干态

总结

# 光场的 Wigner 特征函数



- 经典物理：相空间
- 系统的密度矩阵和相空间的概率分布是否有关系？
- 量子力学：??

$$W(x, p)$$

## 光场的 Wigner 特征函数

- 位置空间概率:

$$\int W(x, p) dp = |\langle x | \psi \rangle|^2$$

- 动量空间概率:

$$\int W(x, p) dx = |\langle p | \psi \rangle|^2$$

## 光场的 Wigner 特征函数

- 位置空间概率:

$$\int W(x, p) dp = \langle x | \rho | x \rangle$$

- 动量空间概率:

$$\int W(x, p) dx = \langle p | \rho | p \rangle$$

## 光场的 Wigner 特征函数：构造 Wigner 函数

- 以位置表象出发:  $\langle x|\psi\rangle\langle\psi|x\rangle$
- 转换到动量表象:

$$\begin{aligned}\langle p|\psi\rangle\langle\psi|p\rangle &= \int \langle p|x_1\rangle\langle x_1|\psi\rangle\langle\psi|x_2\rangle\langle x_2|p\rangle dx_1 dx_2 \\ &= \frac{1}{2\pi\hbar} \int e^{i(x_2-x_1)p/\hbar} \langle x_1|\psi\rangle\langle\psi|x_2\rangle dx_1 dx_2 \\ &= \frac{1}{2\pi\hbar} \int e^{-iyp/\hbar} \left\langle x + \frac{y}{2} \middle| \psi \right\rangle \left\langle \psi \middle| x - \frac{y}{2} \right\rangle dx dy \\ &= \int \left( \frac{1}{2\pi\hbar} \int \left\langle x + \frac{y}{2} \middle| \psi \right\rangle \left\langle \psi \middle| x - \frac{y}{2} \right\rangle e^{-iyp/\hbar} dy \right) dx\end{aligned}$$

- $x_1 = x + \frac{y}{2}, x_2 = x - \frac{y}{2}$
- $\langle x|p\rangle = e^{ixp/\hbar}$

## 光场的 Wigner 特征函数

$$W(x, p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \left\langle x + \frac{1}{2}y \left| \rho \right| x - \frac{1}{2}y \right\rangle e^{-ipy/\hbar} dy$$

也可以定义在动量表象

$$W(x, p) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \left\langle p + \frac{y}{2} \left| \rho \right| p - \frac{y}{2} \right\rangle e^{-ixy/\hbar} dy$$

## 光场的 Wigner 特征函数

$$\begin{aligned}W(x, p) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu \left\langle x - \frac{1}{2}\hbar\nu \left| \rho \right| x + \frac{1}{2}\hbar\nu \right\rangle e^{i p \nu} \\&= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d\nu dx' \langle x' | \rho | x' + \hbar\nu \rangle (2\pi)\delta(x - x' - \hbar\nu/2) e^{i p \nu} \\&= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} du d\nu dx' \langle x' | \rho | x' + \hbar\nu \rangle e^{iu(x-x'-\hbar\nu/2)} e^{i p \nu} \\&= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} du d\nu dx' \langle x' | \rho e^{-iu\hat{x}} | x' + \hbar\nu \rangle e^{iu(x+\hbar\nu/2)} e^{i p \nu} \\&= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} du d\nu dx' \langle x' | \rho e^{-iu\hat{x}} e^{-i p \nu} | x' \rangle e^{iu(x+\hbar\nu/2)} e^{i p \nu} \\&= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} du d\nu dx' \langle x' | \rho e^{-iu\hat{x}} e^{-i p \nu} | x' \rangle e^{i\hbar u \nu / 2} e^{i(u x + \nu p)} \\&= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} du d\nu dx' \langle x' | \rho e^{-iu\hat{x} - i\nu\hat{p}} | x' \rangle e^{i(u x + \nu p)} \\&= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} du d\nu \text{Tr} \left( \rho e^{-iu\hat{x} - i\nu\hat{p}} \right) e^{i(u x + \nu p)}\end{aligned}$$

## 光场的 Wigner 特征函数

把  $\hat{x} = x_{\text{zpf}}(\hat{a} + \hat{a}^\dagger)$ ,  $\hat{p} = \frac{\hbar}{2ix_{\text{zpf}}}(\hat{a} - \hat{a}^\dagger)$  代入

$$\begin{aligned} W(x, p) &= \frac{1}{(2\pi)^2} \int du d\nu \text{Tr} \left( \rho e^{-i(\eta \hat{a}^\dagger + \eta^* \hat{a})} \right) e^{i(ux + \nu p)} \\ &= \frac{1}{(2\pi)^2} \int du d\nu \text{Tr} \left( \rho e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}} \right) e^{i(ux + \nu p)} \\ &= \frac{1}{(2\pi)^2} \int du d\nu \text{Tr} \left( \rho \hat{D}(\lambda) \right) e^{i(ux + \nu p)} \end{aligned}$$

其中  $\eta = x_{\text{zpf}}u + i\frac{\hbar\nu}{2x_{\text{zpf}}}$ ,  $\lambda = -i\eta$

- 特征函数

$$\chi(\lambda, \lambda^*) = \text{Tr} \left( \rho e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}} \right)$$

做傅立叶变换

## 光场的 Wigner 特征函数

$$W(\alpha, \alpha^*) = \frac{1}{2\hbar\pi^2} \int d^2\lambda \chi(\lambda, \lambda^*) e^{-\lambda\alpha^* + \lambda^*\alpha}$$

- 特征函数  $\chi(\lambda, \lambda^*) = \text{Tr}[\rho \hat{D}(\lambda)]$
- 边缘分布

$$\int dp W(x, p) = \langle x | \rho | x \rangle, \quad \int dx W(x, p) = \langle p | \rho | p \rangle,$$

- $|W(x, p)| \leq \frac{1}{\pi\hbar}$

## 光场的 Wigner 特征函数

Weyl 变换：对于任意算符

$$\tilde{A}(x, p) = \int_{-\infty}^{\infty} dy \left\langle x + \frac{1}{2}y \left| \hat{A} \right| x - \frac{1}{2}y \right\rangle e^{-ipy/\hbar}$$

- Wigner 函数是密度算符的 Weyl 变换： $W(x, p) = \tilde{\rho}/(2\pi\hbar)$
- 算符平均值

$$\langle \hat{A} \rangle = \text{Tr}(\rho \hat{A}) = \int_{-\infty}^{\infty} dx dp W(x, p) \tilde{A}(x, p)$$

## 光场的 Wigner 特征函数

Weyl 变换：对于任意算符

$$\tilde{A}(x, p) = \int_{-\infty}^{\infty} dy \left\langle x + \frac{1}{2}y \left| \hat{A} \right| x - \frac{1}{2}y \right\rangle e^{-ipy/\hbar}$$

- 单位算符  $\hat{I}$

$$\langle \hat{I} \rangle = \text{Tr}(\rho \hat{I}) = \int dx dp W_{\rho}(x, p) = 1$$

- 两个密度矩阵  $\rho_1$  和  $\rho_2$

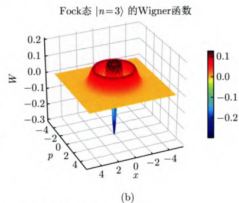
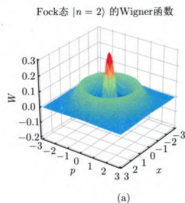
$$\text{Tr}(\rho_1 \rho_2) = 2\pi\hbar \int dx dp W_{\rho_1}(x, p) W_{\rho_2}(x, p)$$

For  $\rho_1 = \rho_2 \rightarrow \text{Tr}(\rho^2) \leq 1$

# 光场的 Wigner 特征函数

例子 1: Fock 态  $\rho = |m\rangle\langle m|$

$$W_m(\alpha) = \frac{2}{\pi} (-1)^n e^{-2|\alpha|^2} L_m(4|\alpha|^2)$$



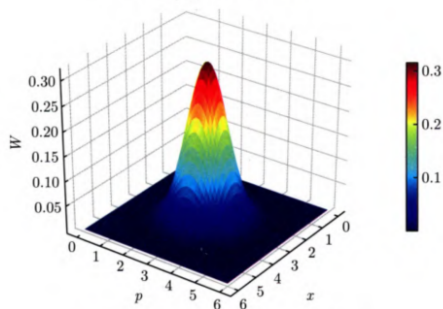
- $L_m(x)$  是  $m$  阶拉盖尔多项式
- $L_m(x)$  有  $m$  个零点

# 光场的 Wigner 特征函数

例子 2: 相干态  $\rho = |\alpha_0\rangle\langle\alpha_0|$

$$W_m(\alpha) = \frac{2}{\pi} e^{-2|\alpha - \alpha_0|^2}$$

相干态  $|\alpha = \frac{3+3i}{\sqrt{2}}\rangle$  的 Wigner 函数



## 光场的 Wigner 特征函数

Wigner 函数小于 0 意味什么？

## 光场的 Wigner 特征函数

Wigner 函数小于 0 意味什么？

- 经典概率不可能出现负数
- 干涉、纠缠、非经典性质

P 表示

$$A(\hat{a}, \hat{a}^\dagger) = \sum_{m,n} c_{mn} (\hat{a}^\dagger)^m \hat{a}^n$$

求  $\hat{A}$  的平均值

$$\begin{aligned}\langle \hat{A} \rangle &= \text{Tr}(\hat{A}\rho) \\ &= \sum_{m,n} c_{mn} \text{Tr}[\rho(\hat{a}^\dagger)^m \hat{a}^n] \\ &= \frac{1}{\pi^2} \int d^2\alpha \sum_{m,n} \text{Tr}[\rho\delta(\alpha^* - \hat{a}^\dagger)\delta(\alpha - \hat{a})] c_{mn}(\alpha^*)^m \alpha^n \\ &= \int d^2\alpha P(\alpha, \alpha^*) A(\alpha, \alpha^*)\end{aligned}$$

P 表示为

$$P(\alpha, \alpha^*) = \text{Tr}[\rho\delta(\alpha^* - \hat{a}^\dagger)\delta(\alpha - \hat{a})]$$

- 也叫 coherent state representation

## P 表示

利用

$$\delta(\alpha^* - \hat{a}^\dagger)\delta(\alpha - \hat{a}) = \frac{1}{\pi^2} \int \exp[-i\lambda(\alpha^* - \hat{a}^\dagger)] \exp[-i\lambda^*(\alpha - \hat{a})] d^2\lambda$$

得到 P 表示

$$P(\alpha, \alpha^*) = \frac{1}{\pi^2} \int d^2\lambda e^{-i\lambda\alpha^* - i\lambda^*\alpha} \text{Tr}(e^{i\lambda\hat{a}^\dagger} e^{i\lambda^*\hat{a}} \rho)$$

## Q 表示

$$A(\hat{a}, \hat{a}^\dagger) = \sum_{m,n} c_{mn} (\hat{a}^\dagger)^m \hat{a}^n$$

求  $\hat{A}$  的平均值

$$\begin{aligned}\langle \hat{A} \rangle &= \text{Tr}(\hat{A}\rho) \\ &= \sum_{m,n} c_{mn} \text{Tr}[\rho(\hat{a}^\dagger)^m \hat{a}^n] \\ &= \frac{1}{\pi^2} \int d^2\alpha \sum_{m,n} \text{Tr}[\rho\delta(\alpha - \hat{a})\delta(\alpha^* - \hat{a}^\dagger)] c_{mn} (\alpha^*)^m \alpha^n \\ &= \int d^2\alpha Q(\alpha, \alpha^*) A(\alpha, \alpha^*)\end{aligned}$$

Q 表示为

$$Q(\alpha, \alpha^*) = \text{Tr}[\rho\delta(\alpha - \hat{a})\delta(\alpha^* - \hat{a}^\dagger)]$$

得到 Q 表示

$$Q(\alpha, \alpha^*) = \frac{1}{\pi^2} \int d^2\lambda e^{-i\lambda\alpha^* - i\lambda^*\alpha} \text{Tr} \left( e^{i\lambda^*\hat{a}} e^{i\lambda\hat{a}^\dagger} \rho \right)$$

也可以回答前面章节介绍的 Husimi-Q 分布函数

$$Q(\alpha, \alpha^*) = \frac{1}{\pi} \int d^2\alpha' \text{Tr} \left[ \rho \delta(\alpha - \hat{a}) |\alpha'\rangle \langle \alpha'| \delta(\alpha^* - \hat{a}^\dagger) \right] = \frac{1}{\pi} \langle \alpha | \rho | \alpha \rangle$$

## 光场相空间的表示

- P 表示:  $\chi_N(\lambda) = \text{Tr}[\rho e^{\lambda \hat{a}^\dagger} e^{-\lambda^* \hat{a}}]$ , 正规算符排序
- Q 表示:  $\chi_N(\lambda) = \text{Tr}[\rho e^{-\lambda^* \hat{a}} e^{\lambda \hat{a}^\dagger}]$ , 反正规算符排序
- W 表示:  $\chi_N(\lambda) = \text{Tr}[\rho e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}}]$ , 对称算符排序

$$\chi_s(\lambda) = \text{Tr}[\rho e^{\lambda \hat{a}^\dagger - \lambda^* \hat{a}} e^{\frac{s}{2} |\lambda|^2}] = \begin{cases} \chi_N, & s = 1 \\ \chi_A, & s = -1 \\ \chi_W, & s = 0 \end{cases}$$

# 光场相空间的表示

性质	P 表示	W 表示	Q 表示
数学性质	可能奇异	平滑	更平滑
是否为正	✗ 可负/奇异	✗ 可负	✓ 总是正
是否像概率	✗	半像	✓ 最像
量子信息保留	完整	完整	有损
平滑程度	最“尖”	中等	最“模糊”

三者都把量子态表示成相空间分布，但在“是否像概率”“是否光滑”“是否包含全部量子信息”之间做了不同取舍。

## 第二章

相干态的性质

压缩相干态

光场的 Wigner 特征函数

自旋相干态/原子相干态

总结

## 自旋相干态/原子相干态

光场与原子相互作用时，如果原子之间的距离远小于光波长，所有原子所感受的光场几乎是一样的

- 只考虑原子内部的二能级，用泡利算符表示： $\sigma_x, \sigma_y, \sigma_z$
- 原子的集体算符

$$\hat{J}_x = \frac{1}{2} \sum_{l=1}^N \sigma_x^{(l)}, \quad \hat{J}_y = \frac{1}{2} \sum_{l=1}^N \sigma_y^{(l)}, \quad \hat{J}_z = \frac{1}{2} \sum_{l=1}^N \sigma_z^{(l)}$$

- 满足角动量对易关系

$$[\hat{J}_x, \hat{J}_y] = i\hat{J}_z, \quad [\hat{J}_y, \hat{J}_z] = i\hat{J}_x, \quad [\hat{J}_z, \hat{J}_x] = i\hat{J}_y$$

- 总角动量算符  $\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$

$$\hat{J}^2 |j, m\rangle = j(j+1) |j, m\rangle, \quad \hat{J}_z |j, m\rangle = m |j, m\rangle$$

- 升降算符

$$\hat{J}_{\pm} = \hat{J}_x \pm i\hat{J}_y$$

## 自旋相干态

由于  $[\hat{J}^2, \hat{J}_+] = [\hat{J}^2, \hat{J}_-] = 0$

$$\hat{J}^2 \hat{J}_+ |j, m\rangle = \hat{J}_+ \hat{J}^2 |j, m\rangle = j(j+1) \hat{J}_+ |j, m\rangle$$

- 如果  $|j, m\rangle$  是  $\hat{J}^2$  本征态, 则  $\hat{J}_+ |j, m\rangle$  也是  $\hat{J}^2$  本征态

## 自旋相干态

由于  $[\hat{J}_z, \hat{J}_+] = \hat{J}_+$

$$\begin{aligned}(\hat{J}_z \hat{J}_+ - \hat{J}_+ \hat{J}_z) |j, m\rangle &= \hat{J}_+ |j, m\rangle \\ \hat{J}_z (\hat{J}_+ |j, m\rangle) &= (m + 1) (\hat{J}_+ |j, m\rangle)\end{aligned}$$

- $\hat{J}_+ |j, m\rangle$  是  $\hat{J}_z$  本征态, 本征值是  $(m + 1)$

$$\hat{J}_+ |j, m\rangle = C_+(j, m) |j, m + 1\rangle$$

- 同样, 利用  $[\hat{J}_z, \hat{J}_-] = -\hat{J}_-$  可以得到

$$\hat{J}_- |j, m\rangle = C_-(j, m) |j, m - 1\rangle$$

## 自旋相干态

以下关系

$$\begin{aligned}\hat{J}_- \hat{J}_+ &= (\hat{J}_x - i\hat{J}_y)(\hat{J}_x + i\hat{J}_y) \\ &= \hat{J}_x^2 + \hat{J}_y^2 + i[\hat{J}_x, \hat{J}_y] = \hat{J}^2 - \hat{J}_z^2 - \hat{J}_z\end{aligned}$$

并有

$$\begin{aligned}\langle j, m | \hat{J}_+^\dagger \hat{J}_+ | j, m \rangle &= C_+^*(j, m) C_+(j, m) \\ \langle j, m | \hat{J}_+^\dagger \hat{J}_+ | j, m \rangle &= \langle j, m | \hat{J}_- \hat{J}_+ | j, m \rangle = j(j+1) - m^2 - m\end{aligned}$$

得到

$$C_+^*(j, m) C_+(j, m) = j(j+1) - m(m+1)$$

可以假定  $C_+(j, m)$  是实数

$$C_+(j, m) = \sqrt{j(j+1) - m(m+1)}$$

## 自旋相干态

Ladder operator

$$\hat{J}_+ |j, m\rangle = \sqrt{j(j+1) - m(m+1)} |j, m+1\rangle$$

$$\hat{J}_- |j, m\rangle = \sqrt{j(j+1) - m(m-1)} |j, m-1\rangle$$

## 自旋相干态

Ladder operator

$$\hat{J}_+ |j, m\rangle = \sqrt{j(j+1) - m(m+1)} |j, m+1\rangle$$

$$\hat{J}_- |j, m\rangle = \sqrt{j(j+1) - m(m-1)} |j, m-1\rangle$$

与玻色子产生算符、湮灭算符对比

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad \hat{a} |n\rangle = \sqrt{n} |n-1\rangle$$

## 自旋相干态

Ladder operator

$$\hat{J}_+ |j, m\rangle = \sqrt{j(j+1) - m(m+1)} |j, m+1\rangle$$

$$\hat{J}_- |j, m\rangle = \sqrt{j(j+1) - m(m-1)} |j, m-1\rangle$$

本征态可以用升降算符表示：

$$\begin{aligned} |j, m\rangle &= \frac{1}{C_{2j}^{j+m}} \frac{(\hat{J}_+)^{j+m}}{(j+m)!} |j, -j\rangle \\ &= \frac{1}{C_{2j}^{j-m}} \frac{(\hat{J}_-)^{j-m}}{(j-m)!} |j, j\rangle \end{aligned}$$

## 自旋相干态

定义自旋相干态为  $N$  个相同单粒子的直积态

$$\begin{aligned} |\theta, \phi\rangle &= \otimes_{l=1}^N \left( \cos \frac{\theta}{2} |0\rangle_l + e^{i\phi} \sin \frac{\theta}{2} |1\rangle_l \right) \\ &= \sum_{m=-j}^j \sqrt{C_{2j}^{j+m}} \left( \cos \frac{\theta}{2} \right)^{j+m} \left( e^{i\phi} \sin \frac{\theta}{2} \right)^{j-m} |j, m\rangle \\ &= \left( \cos \frac{\theta}{2} \right)^{2j} \sum_{m=-j}^j \sqrt{C_{2j}^{j+m}} \left( e^{i\phi} \tan \frac{\theta}{2} \right)^{j-m} |j, m\rangle \\ &= \left( \frac{1}{1 + |\eta|^2} \right)^j \sum_{m=-j}^j \sqrt{C_{2j}^{j+m}} \eta^{j-m} |j, m\rangle \end{aligned}$$

- 设  $\eta = e^{i\phi} \tan \frac{\theta}{2}$

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- 可以证明  $\hat{J}_- |\theta, \phi\rangle = \eta |\theta, \phi\rangle$

## 自旋相干态

- 非正交性
- 超完备性
- 自旋算符的平均值

$$\langle \theta, \phi | \hat{J} | \theta, \phi \rangle = \frac{1}{2} N \mathbf{n} = \frac{1}{2} (\sin \theta \cos \phi, \sin \theta, \cos \theta)$$

## 第二章

相干态的性质

压缩相干态

光场的 Wigner 特征函数

自旋相干态/原子相干态

总结

## Summary

- 相干态、以其性质
- 最小不确定态、压缩相干态
- Wigner 函数、相空间表示
- 自旋相干态