A simple introduction of quantum transport and Hall effects

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Outline

What is transport in condesned matter physics

- Why transport is interesting
- Classical and quantum view of transport

Members of Hall effect and modern physics

- Ordinary Hall effect
- Anomalous Hall effect
- Quantum Hall effect
- Nonlinear version of Hall effect

Summary

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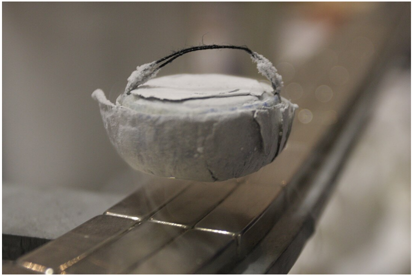
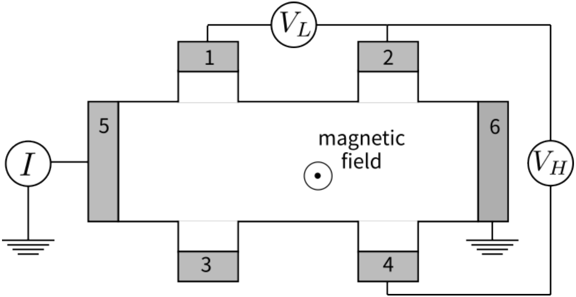
Summary

What is Transport

- Conductivity in metals
- Hall effect/Anomalous Hall effect
- Magnetoresistant
- Superconductor
- Thermoelectric effect
- Spin accumulation and spintronics
- ...

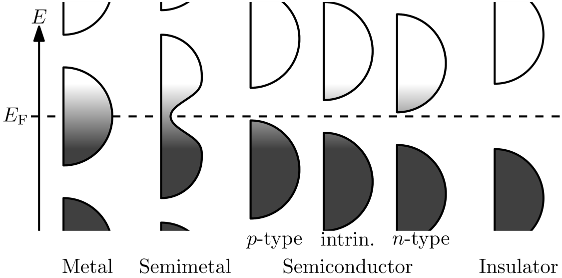


Why we are interested in transport



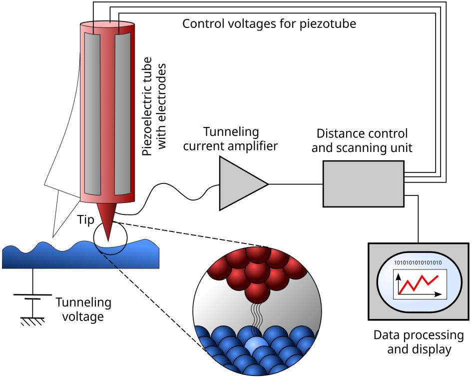
- Material characterization
- Electronic devices
- Sensing devices
- Storage & Calculation
- Information conversion

Why we are interested in transport



- Energy band structure
- Bahavior of the conductivity for metal/semiconductor/insulator with increasing temperature?

Why we are interested in transport

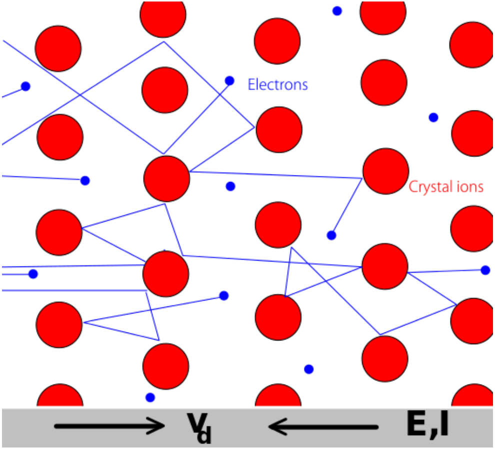


Schematic view of a Scanning tunneling microscope

From wikipedia

- Invented in 1981
- resolution $\sim 0.1\,$nm

Classical view



Classical view

- Newton's second law

$$a = \frac{F}{m_e^*} = -\frac{eE}{m_e^*}$$

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- Between two collisions, the electrons are accelerating. The drift velocity is therefore:

$$v_d = a\tau = -\frac{eE\tau}{m_e^*},$$

  where τ is the mean free time
- The current carried by each electron must be $-ev_d$

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  where τ is the mean free time
- The current carried by each electron must be $-ev_d$
- The current density

$$\mathbf{j} = n(-e)v_d,$$

  where n denotes the carrier density
- The conductivity

$$\sigma = \frac{ne^2\tau}{m_e^*}$$

Classical view

- No information of band structure
- Can not distinguish metal and insulator

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We need quantum methods.

Wavepackets

- Equations of motion for Wavepacket → particle-like behavior (set $e = \hbar = 1$)

$$\dot{\mathbf{r}} = \frac{\partial \varepsilon}{\partial \mathbf{k}} - \dot{\mathbf{k}} \times \mathbf{\Omega}$$
$$\dot{\mathbf{k}} = -\mathbf{E} - \dot{\mathbf{r}} \times \mathbf{B}$$

- Interband information →Incorporate wave-like nature
- Boltzmann equation

$$\frac{\partial f}{\partial t} + \dot{\mathbf{r}} \cdot \frac{\partial f}{\partial \mathbf{r}} + \dot{\mathbf{k}} \cdot \frac{\partial f}{\partial \mathbf{k}} = \mathcal{I}_{\text{coll}}$$

- The current can be calculated by summary each current in every states of the phase space in first Brillouon zone

$$\mathbf{J} = \frac{1}{(2\pi)^{2d}} \sum_n \iint \mathrm{d}\mathbf{k}\, \mathrm{d}\mathbf{r}\, \dot{\mathbf{r}} f(\varepsilon_n)$$

Wavepackets

Relaxation-time approximation

$$\frac{\partial f}{\partial t} + \dot{\mathbf{r}} \cdot \frac{\partial f}{\partial \mathbf{r}} + \dot{\mathbf{k}} \cdot \frac{\partial f}{\partial \mathbf{k}} = -\frac{f - f_0}{\tau}$$

Wavepackets

$$-\mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{k}} = -\frac{f - f_0}{\tau}$$

We obtain an iterative equation for f:

$$f = f_0 + \tau \mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{k}}$$

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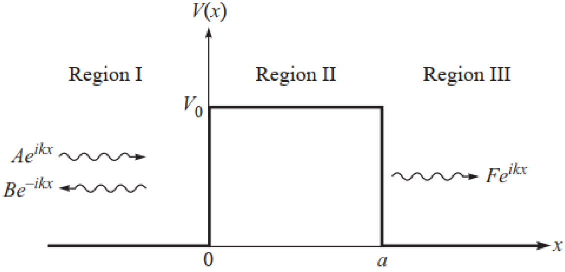
Thus the current density

$$\mathbf{j} = -\frac{1}{(2\pi)^d} \int \mathrm{d}\mathbf{k} \left(\frac{\partial \varepsilon_n}{\partial \mathbf{k}} \right) \tau \mathbf{E} \cdot \frac{\partial f_0}{\partial \mathbf{k}}$$

Square barrier



$$v(x) = \begin{cases} 0 & x < 0 \\ V_0 & 0 < x < a \\ 0 & x > a \end{cases}$$

Square barrier

For $E > V_0$:

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Psi(x) = E\Psi(x)$$

The solution of the Schrödinger equation ($k^2 = 2mE/\hbar^2$)

$$\Psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < 0 \\ Fe^{ikx} + Ge^{-ikx} & x > 0 \end{cases}$$

Square barrier

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- Ae^{ikx} incident plane wave
- Be^{-ikx} reflected plane wave
- Fe^{ikx} transmitted plane wave
- $G = 0$

Square barrier

In region II:

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Psi(x) + V_0\Psi(x) = E\Psi(x)$$

Square barrier

In region II:

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} + k_1^2\Psi = 0, \qquad \text{where} \quad k_1^2 = \frac{2m(E - V_0)}{\hbar^2}$$

Square barrier

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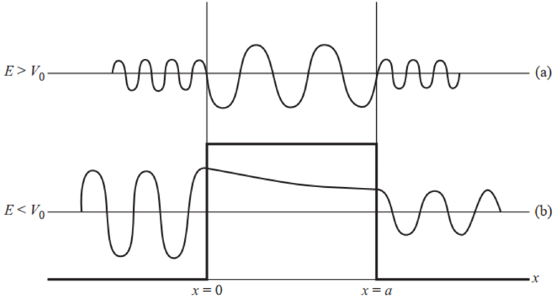
Then the wavefunction is:

$$\Psi(x) \begin{cases} Ae^{ikx} + Be^{-ikx} & x < 0 \\ Ce^{ik_1x} + De^{-ik_1x} & 0 < x < a \\ Fe^{ikx} & x > a \end{cases}$$

Square barrier

In region II:



The wavefunction $\Psi(x)$ and $\frac{\mathrm{d}\Psi}{\mathrm{d}x}$ should be continous at $x = 0$ and $x = a$:

$$A + B = C + D, \quad ik(A - B) = ik_1(C - D)$$

$$Ce^{ik_1 a} + De^{-ik_1 a} = Fe^{ika}, \quad ik_1\left(Ce^{ik_1 a} - De^{-ik_1 a}\right) = ikFe^{ika}$$

Square barrier

The coefficients have the following relations:

$$\frac{B}{A} = \frac{(k^2 - k_1^2)(1 - e^{2ik_1 a})}{(k + k_1)^2 - (k - k_1)^2 e^{2ik_1 a}}$$

$$\frac{F}{A} = \frac{4kk_1 e^{i(k_1 - k)a}}{(k + k_1)^2 - (k - k_1)^2 e^{2ik_1 a}}$$

The reflection and transmission coefficient are

$$R = \left|\frac{B}{A}\right|^2 = \left[1 + \frac{4E(E_0 - V_0)}{V_0^2 \sin^2(k_1 a)}\right]^{-1}$$

$$T = \left|\frac{F}{A}\right|^2 = \left[1 + \frac{V_0^2 \sin^2(k_1 a)}{4E(E - V_0)}\right]^{-1}$$

- Note that if $E < V_0$, k_1 is an imaginary number

Kubo formula

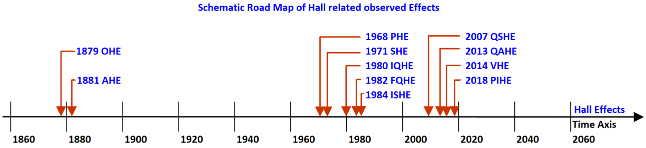
$$\sigma_{xy}^{I} = \frac{e^2}{2\pi V} \mathrm{Tr} \left\langle v_x G^R(\varepsilon_F) v_y G^A(\varepsilon_F) \right\rangle_c$$

- $G^{R/A}$ represent the retarded/advanced Green function

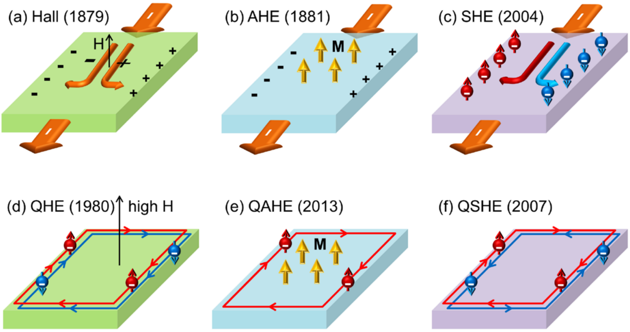
We will not discuss this methods in this talk

Outline

Hall effect


Schematic Road Map of Hall related observed Effects
1879 OHE
1881 AHE
1968 PHE
1971 SHE
1980 IQHE
1982 FQHE
1984 ISHE
2007 QSHE
2013 QAHE
2014 VHE
2018 PIHE
Hall Effects
Time Axis
1860
1880
1900
1920
1940
1960
1980
2000
2020
2040
2060

Members of Hall effect

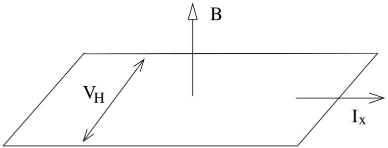


C.-Z. Chang et al., J. Phys.: Condens. Matter 28, 123002 (2016)

Hall effect

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -e\mathbf{E} - e\mathbf{v} \times \mathbf{B} - \frac{m\mathbf{v}}{\tau}$$

Hall effect



Hall effect

$$v_x + \frac{e\tau}{m} v_y B = -\frac{e\tau}{m} E_x$$
$$v_y - \frac{e\tau}{m} v_x B = -\frac{e\tau}{m} E_y$$

Hall effect

The current can be expressed as: $\mathbf{j} = -ne\mathbf{v}$

$$\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix}\mathbf{j} = \frac{e^2 n\tau}{m}\mathbf{E}$$

Hall effect

Given the Ohm's law $\mathbf{j} = \boldsymbol{\sigma}\mathbf{E}$, the conductivity tensor can be obtained

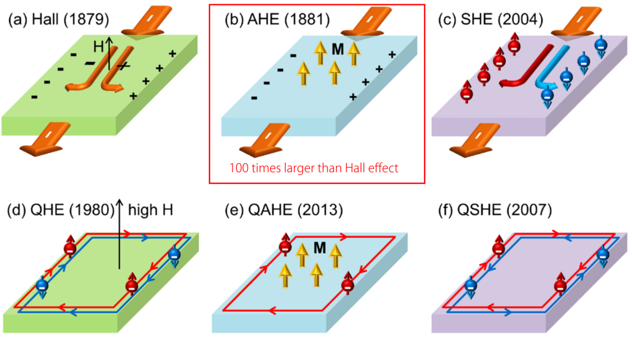
$$\boldsymbol{\sigma} = \frac{\sigma_0}{1+\omega_c^2\tau^2}\begin{pmatrix} 1 & -\omega_c\tau \\ \omega_c\tau & 1 \end{pmatrix}$$

$$\boldsymbol{\rho} = \frac{1}{\sigma_0}\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix}$$

- $\rho_{xy} = \omega_c\tau/\sigma_0 = \frac{B}{ne} = \frac{B}{ne}$, independent of τ
- $\sigma_{xy} \propto n$, linear to carrier density
- $\sigma_{xy} = -\sigma_{yx}$, dissipationless

$$\mathbf{j}^{\text{Hall}} = \begin{pmatrix} 0 & \sigma_{xy} \\ \sigma_{yx} & 0 \end{pmatrix}\mathbf{E} \rightarrow \mathbf{j}^{\text{Hall}} \cdot \mathbf{E} = 0$$

Anomalous Hall effect



- OHE: 1879, Edwin Hall
- AHE: 1881, Edwin Hall

Anomalous Hall effect

- Edwin Hall (1881)
  often very larger than ordinary Hall effect
- Karplus and Luttinger (1954)

Anomalous velocity

- Concept of Berry phase (1984)
- Connect topology and geometry concept to macroscopic current

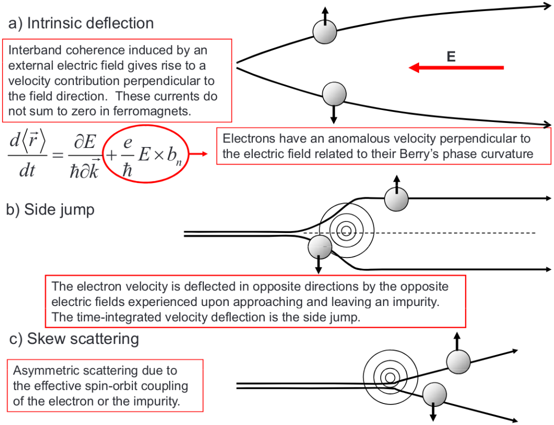
$$\sigma_{ij} = -\varepsilon_{ijk}\frac{e^2}{\hbar}\sum_n \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d}\Omega_n(\mathbf{k})f(\varepsilon_n)$$

Jungwirth, Niu, and MacDonald, 2002

Onoda and Nagaosa, 2002

Haldane, 2004

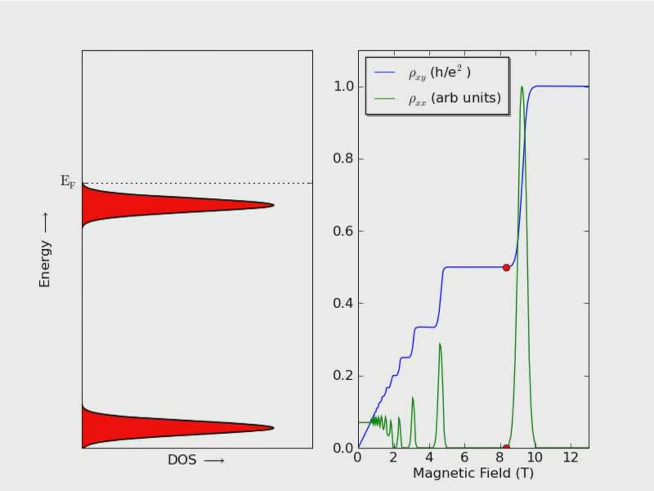
Parsing Anomalous Hall effect



Nagaosa et al., Rev. Mod. Phys. 82, 1539 (2010)

- Skew-scattering (J. Smit, 1950s)
- Side jump (L. Berger, 1970s)

Quantum Hall effect



- 1980, Klaus von Klitzing

Landau levels

Select Landau gauge

$$\mathbf{A}(x, y) = By\widehat{\mathbf{x}}$$

The Hamiltonian in 2D:

$$H = \frac{(\mathbf{p} - \mathbf{eA})^2}{2m} = \frac{(p_x - eBy)^2}{2m} + \frac{p_y^2}{2m}$$
$$= \frac{p_y^2}{2m} + \frac{1}{2}m\left(\frac{eB}{m}\right)^2\left(y - \frac{p_x}{eB}\right)^2$$

Landau levels

Compare with Hamiltonian of harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \qquad\qquad H = \frac{p_y^2}{2m} + \frac{1}{2}m\left(\frac{eB}{m}\right)^2\left(y - \frac{p_x}{eB}\right)^2$$

- The eigenstates is quantized

$$\varepsilon_n = \omega_c\left(n + \frac{1}{2}\right) \quad \text{with} \quad \omega_c = \frac{eB}{m}$$

- The wavefunction is localized in y direction, with a center at $y_0 = \frac{p_x}{eB}$
- In x direction, the wavefunction is a plane wave $\rightarrow$ degeneracy

Landau levels

Degeneracy of Landau levels

- Assume we consider a system with size: $L_x \times L_y$
- The wave-vector along x direction $k_x = \frac{2\pi n}{L_x}$
- The center of the plane wave along y direction must satisty

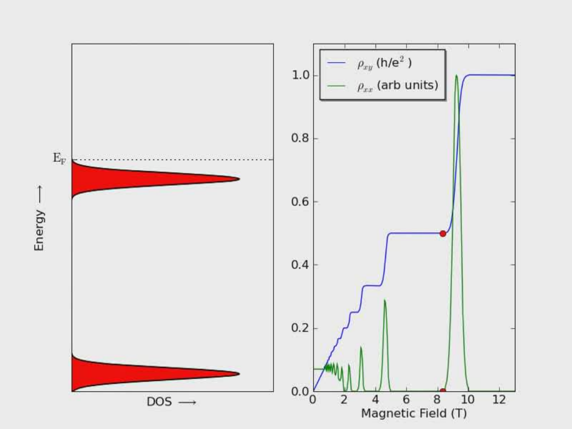
$$0 \leq y_0 \leq L_y \Longrightarrow \frac{\hbar}{eB}\frac{2\pi n}{eB} \leq L_y \Longrightarrow n \leq \frac{L_x L_y eB}{2\pi\hbar}$$

- The maximum number of particles per Landau level is (degeneracy)

$$g = \frac{L_x L_y B}{h/e} = \frac{\Phi}{\Phi_0}$$

without considering the spin degree of freedom

Quantum Hall effect

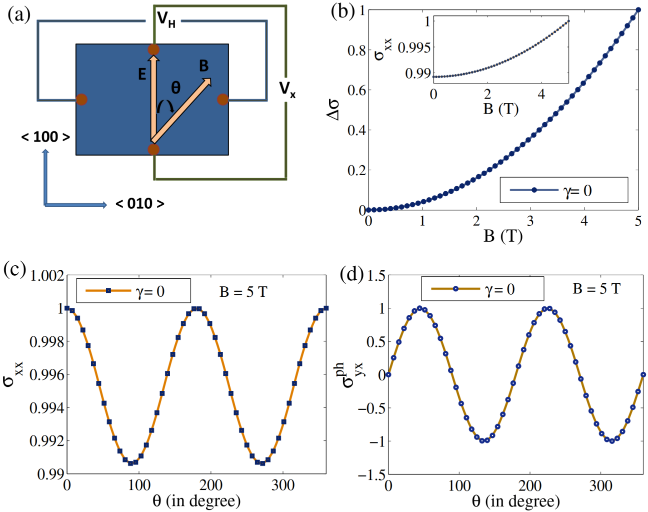


- The density $n = \nu \frac{\Phi/\Phi_0}{S} = \nu \frac{B}{\Phi_0}$
- From Drude model

$$\rho_{xy} = \frac{B}{ne} = \frac{1}{\nu} \frac{h}{e^2}$$

ν Landau levels filled

Planar Hall effect



Nandy et al., Phys. Rev. Lett. 119, 176804 (2017)

Nonlinear Hall effect in non-magnetic system

Outline

Summary